



SpeedLabs

MATHS

CBSE 12th

TEEVRA EDUTECH PVT. LTD.

Application of Derivatives

Exercise 6.3

1. Find the slope of the tangent to the curve $y = 3x^4 - 4x$ at $x = 4$.

Sol. The given curve is $y = 3x^4 - 4x$.

Then, the slope of the tangent to the given curve at $x = 4$ is given by,

$$\left. \frac{dy}{dx} \right]_{x=4} = 12x^3 - 4 \Big]_{x=4} = 12(4)^3 - 4 = 12(64) - 4 = 764$$

2. Find the slope of the tangent to the curve $y = \frac{x-1}{x-2}$, $x \neq 2$ at $x = 10$.

Sol. The given curve is $y = \frac{x-1}{x-2}$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{(x-2)(1) - (x-1)(1)}{(x-2)^2} \\ &= \frac{(x-2-x+1)}{(x-2)^2} = \frac{1}{(x-2)^2} \end{aligned}$$

Thus, the slope of the tangent at $x = 10$ is given by,

$$\left. \frac{dy}{dx} \right]_{x=10} = \left. \frac{-1}{(x-2)^2} \right]_{x=10} = \frac{-1}{(10-2)^2} = \frac{-1}{64}$$

Hence, the slope of the tangent at $x = 10$ is $\frac{-1}{64}$.

3. Find the slope of the tangent to curve $y = x^3 - x + 1$ at the point whose x-coordinate is

Sol. The given curve is $y = x^3 - x + 1$

$$\therefore \frac{dy}{dx} = 3x^2 - 1$$

The slope of the tangent to a curve at (x_0, y_0) is $\left. \frac{dy}{dx} \right]_{(x_0, y_0)}$.

It is given that $x_0 = 2$.

Hence, the slope of the tangent at the point where the x-coordinate is 2 is given by,

$$\left. \frac{dy}{dx} \right]_{x=2} = 3x^2 - 1 \Big]_{x=2} = 3(2)^2 - 1 = 12 - 1 = 11$$

4. Find the slope of the tangent to the curve $y = x^3 - 3x + 2$ at the point whose x-coordinate is 3.

Sol. The given curve is $y = x^3 - 3x + 2$

$$\therefore \frac{dy}{dx} = 3x^2 - 3$$

The slope of the tangent to a curve at (x_0, y_0) is $\left. \frac{dy}{dx} \right|_{(x_0, y_0)}$.

Hence, the slope of the tangent at the point where the x-coordinate is 3 is given by,

$$\left. \frac{dy}{dx} \right|_{x=3} = 3x^2 - 3 \Big|_{x=3} = 3(3)^2 - 3 = 27 - 3 = 24$$

5. Find the slope of the normal to the curve $x = a \cos^3 \theta, y = a \sin^3 \theta$ at $\theta = \frac{\pi}{4}$.

Sol. It is given that $x = a \cos^3 \theta, y = a \sin^3 \theta$.

$$\therefore \frac{dx}{d\theta} = 3a \cos^2 \theta (-\sin \theta) = -3a \cos^2 \theta \sin \theta$$

$$\frac{dy}{d\theta} = 3a \sin^2 \theta (\cos \theta)$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta} \right)}{\left(\frac{dx}{d\theta} \right)} = \frac{3a \sin^2 \theta \cos \theta}{-3a \cos^2 \theta \sin \theta} = -\frac{\sin \theta}{\cos \theta} = -\tan \theta$$

Therefore, the slope of the tangent at $\theta = \frac{\pi}{4}$ is given by,

$$\left. \frac{dy}{dx} \right|_{\theta=\frac{\pi}{4}} = -\tan \theta \Big|_{\theta=\frac{\pi}{4}} = -\tan \frac{\pi}{4} = -1$$

Hence, the slope of the normal at $\theta = \frac{\pi}{4}$ is given by,

$$\frac{1}{\text{slope of the tangent at } \theta = \frac{\pi}{4}} = \frac{-1}{-1} = 1$$

6. Find the slope of the normal to the curve $x = 1 - a \sin \theta, y = b \cos^2 \theta$ at $\theta = \frac{\pi}{2}$.

Sol. It is given that $x = 1 - a \sin \theta$ and $y = b \cos^2 \theta$.

$$\therefore \frac{dx}{d\theta} = -a \cos \theta \text{ and } \frac{dy}{d\theta} = 2b \cos \theta (-\sin \theta) = -2b \sin \theta \cos \theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{-2b \sin \theta \cos \theta}{-a \cos \theta} = \frac{2b}{a} \sin \theta$$

Therefore, the slope of the tangent at $\theta = \frac{\pi}{2}$ is given by,

$$\left. \frac{dy}{dx} \right|_{\theta = \frac{\pi}{2}} = \frac{2b}{a} \sin \theta \Big|_{\theta = \frac{\pi}{2}} = \frac{2b}{a} \sin \frac{\pi}{2} = \frac{2b}{a}$$

Hence, the slope of the normal at $\theta = \frac{\pi}{2}$ is given by,

$$\frac{1}{\text{slope of the tangent at } \theta = \frac{\pi}{2}} = \frac{-1}{\left(\frac{2b}{a}\right)} = \frac{a}{2b}$$

7. Find points at which the tangent to the curve $y = x^3 - 3x^2 - 9x + 7$ is parallel to the x-axis.

Sol. The equation of the given curve is $y = x^3 - 3x^2 - 9x + 7$

$$\therefore \frac{dy}{dx} = 3x^2 - 6x - 9$$

Now, the tangent is parallel to the x-axis if the slope of the tangent is zero.

$$\therefore 3x^2 - 6x - 9 = 0 \Rightarrow x^2 - 2x - 3 = 0$$

$$\Rightarrow (x-3)(x+1) = 0$$

$$\Rightarrow x = 3 \text{ or } x = -1$$

When $x = 3$, $y = (3)^3 - 3(3)^2 - 9(3) + 7 = 27 - 27 - 27 + 7 = -20$.

When $x = -1$, $y = (-1)^3 - 3(-1)^2 - 9(-1) + 7 = -1 - 3 + 9 + 7 = 12$.

Hence, the points at which the tangent is parallel to the x-axis are $(3, -20)$ and $(-1, 12)$.

8. Find a point on the curve $y = (x - 2)^2$ at which the tangent is parallel to the chord joining the points $(2, 0)$ and $(4, 4)$.

Sol. If a tangent is parallel to the chord joining the points $(2, 0)$ and $(4, 4)$, then the slope of the tangent = the slope of the chord.

$$\text{The slope of the chord is } \frac{4-0}{4-2} = \frac{4}{2} = 2$$

Now, the slope of the tangent to the given curve at a point (x, y) is given by,

$$\frac{dy}{dx} = 2(x-2)$$

Since the slope of the tangent = slope of the chord, we have:

$$2(x-2) = 2$$

$$\Rightarrow x-2=1 \Rightarrow x=3$$

$$\text{When } x=3, y=(3-2)^2 = 1$$

Hence, the required point is (3, 1).

9. Find the point on the curve $y = x^3 - 11x + 5$ at which the tangent is $y = x - 11$.

Sol. The equation of the given curve is $y = x^3 - 11x + 5$.

The equation of the tangent to the given curve is given as $y = x - 11$ (which is of the form $y = mx + c$).

\therefore Slope of the tangent = 1

Now, the slope of the tangent to the given curve at the point (x, y) is given by,

$$\frac{dy}{dx} = 3x^2 - 11$$

Then, we have:

$$3x^2 - 11 = 1$$

$$\Rightarrow 3x^2 = 12$$

$$\Rightarrow x^2 = 4$$

$$\Rightarrow x = \pm 2$$

$$\text{When } x = 2, y = (2)^3 - 11(2) + 5 = 8 - 22 + 5 = -9.$$

$$\text{When } x = -2, y = (-2)^3 - 11(-2) + 5 = -8 + 22 + 5 = 19.$$

Hence, the required points are (2, -9) and (-2, 19).

10. Find the equation of all lines having slope -1 that are tangents to the curve $y = \frac{1}{x-1}$, $x \neq 1$.

Sol. The equation of the given curve is $y = \frac{1}{x-1}$, $x \neq 1$.

The slope of the tangents to the given curve at any point (x, y) is given by, $\frac{dy}{dx} = \frac{-1}{(x-1)^2}$

If the slope of the tangent is -1 , then we have:

$$\frac{-1}{(x-1)^2} = -1$$

$$\Rightarrow (x-1)^2 = 1$$

$$x - 1 = \pm 1$$

$$x = 2, 0$$

When $x = 0$, $y = -1$ and when $x = 2$, $y = 1$.

Thus, there are two tangents to the given curve having slope -1 . These are passing through the points $(0, -1)$ and $(2, 1)$.

\therefore The equation of the tangent through $(0, -1)$ is given by,

$$y - (-1) = -1(x - 0)$$

$$\Rightarrow y + 1 = -x$$

$$\Rightarrow y + x + 1 = 0$$

\therefore The equation of the tangent through $(2, 1)$ is given by,

$$y - 1 = -1(x - 2)$$

$$\Rightarrow y - 1 = -x + 2$$

$$\Rightarrow y + x - 3 = 0$$

Hence, the equations of the required lines are $y + x + 1 = 0$ and $y + x - 3 = 0$.

11. Find the equation of all lines having slope 2 which are tangents to the curve $y = \frac{1}{x-3}$, $x \neq 3$.

Sol. The equation of the given curve is $y = \frac{1}{x-3}$, $x \neq 3$.

The slope of the tangent to the given curve at any point (x, y) is given by, $\frac{dy}{dx} = \frac{-1}{(x-3)^2}$.

If the slope of the tangent is 2, then we have:

$$\frac{-1}{(x-3)^2} = 2$$

$$\Rightarrow 2(x-3)^2 = -1$$

$$\Rightarrow (x-3)^2 = \frac{-1}{2}$$

This is not possible since the L.H.S. is positive while the R.H.S. is negative.

Hence, there is no tangent to the given curve having slope 2.

12. Find the equations of all lines having slope 0 which are tangent to the curve $y = \frac{1}{x^2 - 2x + 3}$.

Sol. The equation of the given curve is $y = \frac{1}{x^2 - 2x + 3}$.

The slope of the tangent to the given curve at any point (x, y) is given by,

$$\frac{dy}{dx} = \frac{-(2x-2)}{(x^2-2x+3)^2} = \frac{-2(x-1)}{(x^2-2x+3)^2}$$

If the slope of the tangent is 0, then we have:

$$\frac{-2(x-1)}{(x^2-2x+3)^2} = 0$$

$$\Rightarrow -2(x-1) = 0$$

$$\Rightarrow x = 1$$

$$\text{When } x = 1, y = \frac{1}{1+2+3} = \frac{1}{2}$$

∴ The equation of the tangent through $\left(1, \frac{1}{2}\right)$ is given by,

$$y - \frac{1}{2} = 0(x-1)$$

$$\Rightarrow y - \frac{1}{2} = 0 \quad \Rightarrow y = \frac{1}{2}$$

13. Find points on the curve $\frac{x^2}{9} + \frac{y^2}{16} = 1$ at which the tangents are

(i) parallel to x-axis (ii) parallel to y-axis

Sol. The equation of the given curve is $\frac{x^2}{9} + \frac{y^2}{16} = 1$.

On differentiating both sides with respect to x, we have:

$$\frac{2x}{9} + \frac{2y}{16} \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-16x}{9y}$$

(i) The tangent is parallel to the x-axis if the slope of the tangent is i.e., $\frac{-16x}{9y} = 0$, which is possible if $x = 0$.

$$\text{Then, } \frac{x^2}{9} + \frac{y^2}{16} = 1 \text{ for } x = 0$$

$$\Rightarrow y^2 = 16 \Rightarrow y = \pm 4$$

Hence, the points at which the tangents are parallel to the x-axis are (0, 4) and (0, -4).

(ii) The tangent is parallel to the y-axis if the slope of the normal is 0, which gives $\frac{-1}{\left(\frac{-16x}{9y}\right)} = \frac{9y}{16x} = 0$

$$\Rightarrow y = 0.$$

$$\text{Then, } \frac{x^2}{9} + \frac{y^2}{16} = 1 \text{ for } y = 0.$$

$$\Rightarrow x = \pm 3$$

Hence, the points at which the tangents are parallel to the y-axis are (3, 0) and (-3, 0).

14. Find the equations of the tangent and normal to the given curves at the indicated points:

(i) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at (0, 5)

(ii) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at (1, 3)

(iii) $y = x^3$ at (1, 1)

(iv) $y = x^2$ at (0, 0)

(v) $x = \cos t, y = \sin t$ at $t = \frac{\pi}{4}$

Sol. (i) The equation of the curve is $y = x^4 - 6x^3 + 13x^2 - 10x + 5$

On differentiating with respect to x, we get:

$$\frac{dy}{dx} = 4x^3 - 18x^2 + 26x - 10$$

$$\left. \frac{dy}{dx} \right|_{(0,5)} = -10$$

Thus, the slope of the tangent at (0, 5) is -10. The equation of the tangent is given as:

$$y - 5 = -10(x - 0)$$

$$\Rightarrow y - 5 = -10x$$

$$\Rightarrow 10x + y = 5$$

The slope of the normal at (0, 5) is $\frac{1}{\text{Slope of the tangent at (0,5)}} = \frac{1}{10}$

Therefore, the equation of the normal at (0, 5) is given as:

$$y - 5 = \frac{1}{10}(x - 0)$$

$$\Rightarrow 10y - 50 = x$$

$$\Rightarrow 10y - 50 = x$$

$$\Rightarrow x - 10y + 50 = 0$$

(ii) The equation of the curve is $y = x^4 - 6x^3 + 13x^2 - 10x + 5$.

On differentiating with respect to x , we get:

$$\frac{dy}{dx} = 4x^3 - 18x^2 + 26x - 10$$

$$\left. \frac{dy}{dx} \right|_{(1,3)} = 4 - 18 + 26 - 10 = 2$$

Thus, the slope of the tangent at $(1, 3)$ is 2. The equation of the tangent is given as:

$$y - 3 = 2(x - 1)$$

$$\Rightarrow y - 3 = 2x - 2$$

$$\Rightarrow y = 2x + 1$$

The slope of the normal at $(1, 3)$ is $\frac{-1}{\text{slope of the tangent at } (1,3)} = \frac{-1}{2}$

Therefore, the equation of the normal at $(1, 3)$ is given as:

$$y - 3 = -\frac{1}{2}(x - 1)$$

$$\Rightarrow 2y - 6 = -x + 1$$

$$\Rightarrow x + 2y - 7 = 0$$

(iii) The equation of the curve is $y = x^3$.

On differentiating with respect to x , we get:

$$\frac{dy}{dx} = 3x^2$$

$$\left. \frac{dy}{dx} \right|_{(1,1)} = 3(1)^2 = 3$$

Thus, the slope of the tangent at $(1, 1)$ is 3 and the equation of the tangent is given as:

$$y - 1 = \frac{-1}{3}(x - 1)$$

$$\Rightarrow 3y - 3 = -x + 1$$

$$\Rightarrow x + 3y - 4 = 0$$

(iv) The equation of the curve is $y = x^2$.

On differentiating with respect to x , we get:

$$\frac{dy}{dx} = 2x$$

$$\left. \frac{dy}{dx} \right|_{(0,0)} = 0$$

Thus, the slope of the tangent at (0, 0) is 0 and the equation of the tangent is given as:

$$y - 0 = 0(x - 0)$$

$$\Rightarrow y = 0$$

The slope of the normal at (0, 0) is $\frac{-1}{\text{slope of the tangent at (0,0)}} = -\frac{1}{0}$, which is not defined.

Therefore, the equation of the normal at $(x_0, y_0) = (0, 0)$ is given by $x = x_0 = 0$

(v) The equation of the curve is $x = \cos t, y = \sin t$.

$$x = \cos t \text{ and } y = \sin t$$

$$\therefore \frac{dx}{dt} = -\sin t, \frac{dy}{dt} = \cos t$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\cos t}{-\sin t} = -\cot t$$

$$\left. \frac{dy}{dx} \right|_{t=\frac{\pi}{4}} = -\cot t = -1$$

The slope of the tangent at $t = \frac{\pi}{4}$ is -1 .

$$\text{When } t = \frac{\pi}{4}, x = \frac{1}{\sqrt{2}} \text{ and } y = \frac{1}{\sqrt{2}}.$$

Thus, the equation of the tangent to the given curve at $t = \frac{\pi}{4}$, i.e., at $\left[\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right]$ is $y - \frac{1}{\sqrt{2}} = -1\left(x - \frac{1}{\sqrt{2}}\right)$

$$\Rightarrow x + y - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0$$

$$\Rightarrow x + y - \sqrt{2} = 0$$

The slope of the normal at $t = \frac{\pi}{4}$ is $\frac{-1}{\text{slope of the tangent at } t = \frac{\pi}{4}} = 1$

Therefore, the equation of the normal to the given curve at $t = \frac{\pi}{4}$, i.e., at $\left[\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right]$ is

$$y - \frac{1}{\sqrt{2}} = -1 \left(x - \frac{1}{\sqrt{2}} \right).$$

$$\Rightarrow x = y.$$

15. Find the equation of the tangent line to the curve $y = x^2 - 2x + 7$ which is
- (a) parallel to the line $2x - y + 9 = 0$
 - (b) perpendicular to the line $5y - 15x = 13$.

Sol. The equation of the given curve is $y = x^2 - 2x + 7$.

On differentiating with respect to x , we get:

$$\frac{dy}{dx} = 2x - 2$$

(a) The equation of the line is $2x - y + 9 = 0$.

$$2x - y + 9 = 0 \therefore y = 2x + 9$$

This is of the form $y = mx + c$.

\therefore Slope of the line = 2

If a tangent is parallel to the line $2x - y + 9 = 0$, then the slope of the tangent is equal to the slope of the line. Therefore, we have:

$$2 = 2x - 2$$

$$\Rightarrow 2x = 4$$

$$\Rightarrow x = 2$$

Now, $x = 2$

$$\Rightarrow y = 4 - 4 + 7 = 7$$

Thus, the equation of the tangent passing through $(2, 7)$ is given by,

$$y - 7 = 2(x - 2)$$

$$\Rightarrow y - 2x - 3 = 0$$

Hence, the equation of the tangent line to the given curve (which is parallel to line $2x - y + 9 = 0$) is $y - 2x - 3 = 0$

(b) The equation of the line is $5y - 15x = 13$.

$$5y - 15x = 13 \therefore y = 3x + \frac{13}{5}$$

This is of the form $y = mx + c$.

\therefore Slope of the line = 3

If a tangent is perpendicular to the line $5y - 15x = 13$, then the slope of the tangent is

$$\frac{-1}{\text{slope of the line}} = \frac{-1}{3}$$

$$\Rightarrow 2x - 2 = \frac{-1}{3}$$

$$\Rightarrow 2x = \frac{-1}{3} + 2 \quad \Rightarrow 2x = \frac{5}{3}$$

$$\Rightarrow x = \frac{5}{6} \quad \text{Now, } x = \frac{5}{6}$$

$$\Rightarrow y = \frac{25}{36} - \frac{10}{6} + 7 = \frac{25 - 60 + 252}{36} = \frac{217}{36}$$

Thus, the equation of the tangent passing through $\left(\frac{5}{6}, \frac{217}{36}\right)$ is given by

$$y - \frac{217}{36} = -\frac{1}{3}\left(x - \frac{5}{6}\right)$$

$$\Rightarrow \frac{36y - 217}{36} = -\frac{1}{18}(6x - 5)$$

$$\Rightarrow 36y - 217 = -2(6x - 5)$$

$$\Rightarrow 36y - 217 = -12x + 10$$

$$\Rightarrow 36y + 12x - 227 = 0$$

Hence, the equation of the tangent line to the given curve (which is perpendicular to line $5y - 15x = 13$) $36y + 12x - 227 = 0$

16. Show that the tangents to the curve $y = 7x^3 + 11$ at the points where $x = 2$ and $x = -2$ are parallel.

Sol. The equation of the given curve is $y = 7x^3 + 11$.

$$\therefore \frac{dy}{dx} = 21x^2$$

The slope of the tangent to a curve at (x_0, y_0) is $\left. \frac{dy}{dx} \right|_{(x_0, y_0)}$

Therefore, the slope of the tangent at the point where $x = 2$ is given by,

$$\left. \frac{dy}{dx} \right|_{x=2} = 21(2)^2 = 84$$

It is observed that the slopes of the tangents at the points where $x = 2$ and $x = -2$ are equal.

Hence, the two tangents are parallel.

17. Find the points on the curve $y = x^3$ at which the slope of the tangent is equal to the y-coordinate of the point.

Sol. The equation of the given curve is $y = x^3$.

$$\therefore \frac{dy}{dx} = 3x^2$$

The slope of the tangent at the point (x, y) is given by,

$$\left. \frac{dy}{dx} \right|_{(x,y)} = 3x^2$$

When the slope of the tangent is equal to the y-coordinate of the point, then $y = 3x^2$.

Also, we have $y = x^3$.

$$\therefore 3x^2 = x^3$$

$$\therefore x^2(x-3) = 0$$

$$\therefore x = 0, x = 3$$

When $x = 0$, then $y = 0$ and when $x = 3$, then $y = 3(3)^2 = 27$.

Hence, the required points are $(0, 0)$ and $(3, 27)$.

18. For the curve $y = 4x^3 - 2x^5$, find all the points at which the tangents pass through the origin.

Sol. The equation of the given curve is $y = 4x^3 - 2x^5$.

$$\therefore \frac{dy}{dx} = 12x^2 - 10x^4$$

Therefore, the slope of the tangent at a point (x, y) is $12x^2 - 10x^4$.

The equation of the tangent at (x, y) is given by,

$$Y - y = (12x^2 - 10x^4)(X - x) \quad \dots\dots(1)$$

When the tangent passes through the origin $(0, 0)$, then $X = Y = 0$.

Therefore, equation (1) reduces to:

$$-y = (12x^2 - 10x^4)(-x)$$

$$y = 12x^3 - 10x^5$$

Also, we have

$$y = 4x^3 - 2x^5$$

$$\therefore 12x^3 - 10x^5 = 4x^3 - 2x^5$$

$$\Rightarrow 8x^3 - 8x^5 = 0$$

$$\Rightarrow x^3 - x^5 = 0$$

$$\Rightarrow x^5 - x^3 = 0$$

$$\Rightarrow x^3(x^2 - 1) = 0$$

$$\Rightarrow x = 0, \pm 1$$

$$\text{When } x = 0, y = 4(0)^3 - 2(0)^5 = 0$$

$$\text{When } x = 1, y = 4(1)^3 - 2(1)^5 = 2.$$

$$\text{When } x = -1, y = 4(-1)^3 - 2(-1)^5 = -2.$$

Hence, the required points are $(0, 0)$, $(1, 2)$, and $(-1, -2)$.

19. Find the points on the curve $x^2 + y^2 - 2x - 3 = 0$ at which the tangents are parallel to the x-axis.

Sol. The equation of the given curve is $x^2 + y^2 - 2x - 3 = 0$.

On differentiating with respect to x , we have:

$$2x + 2y \frac{dy}{dx} - 2 = 0$$

$$\Rightarrow y \frac{dy}{dx} = 1 - x$$

$$\Rightarrow \frac{dy}{dx} = \frac{1-x}{y}$$

Now, the tangents are parallel to the x-axis if the slope of the tangent is 0.

$$\therefore \frac{1-x}{y} = 0 \Rightarrow 1-x = 0 \Rightarrow x = 1$$

But, $x^2 + y^2 - 2x - 3 = 0$ for $x = 1$.

$$\Rightarrow y^2 = 4 \therefore y = \pm 2$$

Hence, the points at which the tangents are parallel to the x-axis are $(1, 2)$ and $(1, -2)$.

20. Find the equation of the normal at the point (am^2, am^3) for the curve $ay^2 = x^3$.

Sol. The equation of the given curve is $ay^2 = x^3$.

On differentiating with respect to x , we have:

$$2ay \frac{dy}{dx} = 3x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{3x^2}{2ay}$$

The slope of a tangent to the curve at (x_0, y_0) is $\left. \frac{dy}{dx} \right|_{(x_0, y_0)}$.

The slope of the tangent to the given curve at (am^2, am^3) is $\left. \frac{dy}{dx} \right|_{(am^2, am^3)} = \frac{3(am^2)^2}{2a(am^3)} = \frac{3a^2m^4}{2a^2m^3} = \frac{3m}{2}$.

\therefore Slope of normal at (am^2, am^3)

$$= \frac{-1}{\text{slope of the tangent at } (am^2, am^3)} = \frac{-2}{3m}$$

Hence, the equation of the normal at (am^2, am^3) is given by,

$$y - am^3 = \frac{-2}{3m}(x - am^2)$$

$$\Rightarrow 3my - 3am^4 = -2x + 2am^2$$

$$\Rightarrow 2x + 3my - am^2(2 + 3m^2) = 0$$

21. Find the equation of the normals to the curve $y = x^3 + 2x + 6$ which are parallel to the line $x + 14y + 4 = 0$.

Sol. The equation of the given curve is $y = x^3 + 2x + 6$.

The slope of the tangent to the given curve at any point (x, y) is given by,

$$\frac{dy}{dx} = 3x^2 + 2$$

Slope of the normal to the given curve at any point (x, y)

$$= \frac{-1}{\text{slope of the tangent at the point } (x, y)}$$

$$= \frac{-1}{3x^2 + 2}$$

The equation of the given line is $x + 14y + 4 = 0$.

$$x + 14y + 4 = 0 \quad \therefore y = -\frac{1}{14}x - \frac{4}{14} \quad (\text{which is of the form } y = mx + c)$$

$$\therefore \text{Slope of the given line} = \frac{-1}{14}$$

If the normal is parallel to the line, then we must have the slope of the normal being equal to the slope of the line.

$$\therefore \frac{-1}{3x^2 + 2} = \frac{-1}{14}$$

$$\Rightarrow 3x^2 + 2 = 14$$

$$\Rightarrow 3x^2 = 12$$

$$\Rightarrow x^2 = 4$$

$$\Rightarrow x = \pm 2$$

$$\text{When } x = 2, y = 8 + 4 + 6 = 18.$$

$$\text{When } x = -2, y = -8 - 4 + 6 = -6.$$

Therefore, there are two normals to the given curve with slope $\frac{-1}{14}$ and passing through

the points $(2, 18)$ and $(-2, -6)$.

Thus, the equation of the normal through $(2, 18)$ is given by,

$$y - 18 = \frac{-1}{14}(x - 2)$$

$$\Rightarrow 14y - 252 = x + 2$$

$$\Rightarrow x + 14y - 254 = 0$$

And, the equation of the normal through $(-2, -6)$ is given by,

$$y - (-6) = \frac{-1}{14}[x - (-2)]$$

$$\Rightarrow y + 6 = \frac{-1}{14}(x + 2)$$

$$\Rightarrow 14y + 84 = -x - 2$$

$$\Rightarrow x + 14y + 86 = 0$$

Hence, the equations of the normals to the given curve (which are parallel to the given line) are $x + 14y - 254 = 0$ and $x + 14y + 86 = 0$.

22. Find the equations of the tangent and normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$.

Sol. The equation of the given parabola is $y^2 = 4ax$.

On differentiating $y^2 = 4ax$ with respect to x , we have:

$$2y \frac{dy}{dx} = 4a$$

$$\Rightarrow \frac{dy}{dx} = \frac{2a}{y}$$

$$\therefore \text{The slope of the tangent at } (at^2, 2at) \text{ is } \left. \frac{dy}{dx} \right|_{(at^2, 2at)} = \frac{2a}{2at} = \frac{1}{t}.$$

Then, the equation of the tangent at $(at^2, 2at)$ is given by,

$$y - 2at = \frac{1}{t}(x - at^2)$$

$$\Rightarrow ty - 2at^2 = x - at^2$$

$$\Rightarrow ty = x + at^2$$

Now, the slope of the normal at $(at^2, 2at)$ is given by,

$$= \frac{-1}{\text{slope of the tangent at } (at^2, 2at)} = -t$$

Thus, the equation of the normal at $(at^2, 2at)$ is given as:

$$y - 2at = -t(x - at^2)$$

$$\Rightarrow y - 2at = -tx + at^3$$

$$\Rightarrow y = -tx + 2at + at^3$$

- 23.** Prove that the curves $x = y^2$ and $xy = k$ cut at right angles if $8k^2 = 1$. [Hint: Two curves intersect at right angle if the tangents to the curves at the point of intersection are perpendicular to each other.]

Sol. The equations of the given curves are given as $x = y^2$ and $xy = k$.

Putting $x = y^2$ in $xy = k$, we get:

$$y^3 = k \Rightarrow y = k^{\frac{1}{3}}$$

$$\therefore x = k^{\frac{2}{3}}$$

Thus, the point of intersection of the given curves is $\left(k^{\frac{2}{3}}, k^{\frac{1}{3}}\right)$.

Differentiating $x = y^2$ with respect to x , we have:

$$1 = 2y \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{2y}$$

Therefore, the slope of the tangent to the curve $x = y^2$ at $\left(k^{\frac{2}{3}}, k^{\frac{1}{3}}\right)$ is $\left. \frac{dy}{dx} \right|_{\left(k^{\frac{2}{3}}, k^{\frac{1}{3}}\right)} = \frac{1}{2k^{\frac{1}{3}}}$.

On differentiating $xy = k$ with respect to x , we have:

$$x \frac{dy}{dx} + y = 0 \Rightarrow \frac{dy}{dx} = \frac{-y}{x}$$

\therefore Slope of the tangent to the curve $xy = k$ at $\left(k^{\frac{2}{3}}, k^{\frac{1}{3}}\right)$ is given by,

$$\left. \frac{dy}{dx} \right|_{\left(k^{\frac{2}{3}}, k^{\frac{1}{3}}\right)} = \left. \frac{-y}{x} \right|_{\left(k^{\frac{2}{3}}, k^{\frac{1}{3}}\right)} = \frac{-k^{\frac{1}{3}}}{k^{\frac{2}{3}}} = \frac{-1}{k^{\frac{1}{3}}}$$

We know that two curves intersect at right angles if the tangents to the curves at the point of intersection i.e., at $\left(k^{\frac{2}{3}}, k^{\frac{1}{3}}\right)$ are perpendicular to each other. This implies that we should have the product of the tangents as -1 .

Thus, the given two curves cut at right angles if the product of the slopes of their respective tangents at $\left(k^{\frac{2}{3}}, k^{\frac{1}{3}}\right)$ is -1 .

$$\text{i.e., } \left(\frac{1}{2k^{\frac{1}{3}}}\right)\left(\frac{-1}{2k^{\frac{1}{3}}}\right) = -1$$

$$\Rightarrow 2k^{\frac{2}{3}} = 1$$

$$\left(2k^{\frac{2}{3}}\right)^3 = (1)^3$$

$$\Rightarrow 8k^2 = 1$$

Hence, the given two curves cut at right angles if $8k^2 = 1$.

24. Find the equations of the tangent and normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_0, y_0) .

Sol. Differentiating $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with respect to x , we have:

$$\frac{2x}{a^2} - \frac{2y}{b^2} \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{2y}{b^2} \frac{dy}{dx} = \frac{2x}{a^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{b^2 x}{a^2 y}$$

Therefore, the slope of the tangent at (x_0, y_0) is $\left. \frac{dy}{dx} \right|_{(x_0, y_0)} = \frac{b^2 x_0}{a^2 y_0}$.

Then, the equation of the tangent at (x_0, y_0) is given by,

$$y - y_0 = \frac{b^2 x_0}{a^2 y_0} (x - x_0)$$

$$\Rightarrow a^2yy_0 - a^2y_0^2 = b^2xx_0 - b^2x_0^2$$

$$\Rightarrow b^2xx_0 - a^2yy_0 - b^2x_0^2 + a^2y_0^2 = 0$$

$$\Rightarrow \frac{xx_0}{a^2} - \frac{yy_0}{b^2} - \left(\frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} \right) = 0 \quad \text{[On dividing both sides by } a^2b^2 \text{]}$$

$$\Rightarrow \frac{xx_0}{a^2} - \frac{yy_0}{b^2} - 1 = 0 \quad \left[(x_0, y_0) \text{ lies on the hyperbola } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \right]$$

$$\Rightarrow \frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1$$

Now, the slope of the normal at (x_0, y_0) is given by,

$$= \frac{-1}{\text{slope of the tangent at } (x_0, y_0)} = \frac{-a^2y_0}{b^2x_0}$$

Hence, the equation of the normal at (x_0, y_0) is given by

$$y - y_0 = \frac{-a^2y_0}{b^2x_0}(x - x_0)$$

$$\Rightarrow \frac{y - y_0}{a^2y_0} = \frac{-(x - x_0)}{b^2x_0}$$

$$\Rightarrow \frac{y - y_0}{a^2y_0} + \frac{(x - x_0)}{b^2x_0} = 0$$

25. Find the equation of the tangent to the curve $y = \sqrt{3x - 2}$ which is parallel to the line $4x - 2y + 5 = 0$.

Sol. The equation of the given curve is $y = \sqrt{3x - 2}$

The slope of the tangent to the given curve at any point (x, y) is given by,

The equation of the given line is $4x - 2y + 5 = 0$.

$$4x - 2y + 5 = 0 \quad \therefore y = 2x + \frac{5}{2} \quad (\text{which is of the form } y = mx + c)$$

\therefore Slope of the line = 2

Now, the tangent to the given curve is parallel to the line $4x - 2y - 5 = 0$ if the slope of the tangent is equal to the slope of the line.

$$\frac{3}{2\sqrt{3x - 2}} = 2$$

$$\Rightarrow \sqrt{3x-2} = \frac{3}{4}$$

$$\Rightarrow 3x-2 = \frac{9}{16}$$

$$\Rightarrow 3x = \frac{9}{16} + 2 = \frac{41}{16}$$

$$\Rightarrow x = \frac{41}{48}$$

$$\text{When } x = \frac{41}{48}, y = \sqrt{3\left(\frac{41}{16}\right) - 2} = \sqrt{\frac{41}{16} - 2} = \sqrt{\frac{41-32}{16}} = \sqrt{\frac{9}{16}} = \frac{3}{4}$$

∴ Equation of the tangent passing through the point $\left(\frac{41}{48}, \frac{3}{4}\right)$ is given by,

$$y - \frac{3}{4} = 2\left(x - \frac{41}{48}\right)$$

$$\Rightarrow \frac{4y-3}{4} = 2\left(\frac{48x-41}{48}\right)$$

$$\Rightarrow 4y-3 = \frac{48x-41}{6}$$

$$\Rightarrow 24y-18 = 48x-41$$

$$\Rightarrow 48x-24y = 23$$

Hence, the equation of the required tangent is $48x - 24y = 23$.

26. The slope of the normal to the curve $y = 2x^2 + 3 \sin x$ at $x = 0$ is

- (A) 3 (B) $\frac{1}{3}$ (C) -3 (D) $-\frac{1}{3}$

Sol. The equation of the given curve is $y = 2x^2 + 3 \sin x$.

Slope of the tangent to the given curve at $x = 0$ is given by,

$$\left. \frac{dy}{dx} \right]_{x=0} = 4x + 3 \cos x \Big]_{x=0} = 0 + 3 \cos 0 = 3$$

Hence, the slope of the normal to the given curve at $x = 0$ is $= \frac{-1}{\text{slope of the tangent at } x=0} = \frac{-1}{3}$

The correct answer is D.

27. The line $y = x + 1$ is a tangent to the curve $y^2 = 4x$ at the point
(A) (1, 2) (B) (2, 1) (C) (1, -2) (D) (-1, 2)

Sol. The equation of the given curve is $y^2 = 4x$.

Differentiating with respect to x , we have:

$$2y \frac{dy}{dx} = 4 \Rightarrow \frac{dy}{dx} = \frac{2}{y}$$

Therefore, the slope of the tangent to the given curve at any point (x, y) is given by,

$$\frac{dy}{dx} = \frac{2}{y}$$

The given line is $y = x + 1$ (which is of the form $y = mx + c$)

\therefore Slope of the line = 1

The line $y = x + 1$ is a tangent to the given curve if the slope of the line is equal to the slope of the tangent.

Also, the line must intersect the curve.

Thus, we must have:

$$\frac{2}{y} = 1$$

$$\Rightarrow y = 2$$

$$\text{Now, } y = x + 1 \Rightarrow x = y - 1 \Rightarrow x = 2 - 1 = 1$$

Hence, the line $y = x + 1$ is a tangent to the given curve at the point (1, 2).

The correct answer is A.