SpeedLabs

MATHS

CBSE 12th

TEEVRA EDUTECH PVT. LTD.

Continuity and Differentiability
Exercise-5.1

Q.1 Prove that the function $f(x) = 5x - 3$ is continuous at $x = 0$, at $x = -3$ and at $x = 5$.

Sol: The given function is $f(x) = 5x - 3$

At $x = 0, f(0) = 5 \times 0 - 3 = -3$

$\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0}(5x - 3) = 5 \times 0 - 3 = -3$

$\therefore \lim\limits_{x\to 0} f(x) = f(0)$

Therefore, f is continuous at $x = 0$

At $x = -3, f(-3) = 5 \times (-3) - 3 = -18$

$\lim\limits_{x\to -3} f(x) = \lim\limits_{x\to -3}(5x - 3) = 5 \times (-3) - 3 = -18$

$\therefore \lim\limits_{x\to -3} f(x) = f(-3)$

Therefore, f is continuous at $x = -3$

At $x = 0, f(0) = 5 \times 5 - 3 = 25 - 3 = 22$

$\lim\limits_{x\to 5} f(x) = \lim\limits_{x\to 5}(5x - 3) = 5 \times 5 - 3 = 22$

$\therefore \lim\limits_{x\to 5} f(x) = f(5)$

Therefore, f is continuous at $x = 5$

Q.2 Examine the continuity of the function $f(x) = 2x^2 - 1$ at $x = 3$

Sol: The given function is $f(x) = 2x^2 - 1$

At $x = 3, f(x) = f(3) = 2 \times 3^2 - 1 = 17$

$\lim\limits_{x\to 3} f(x) = \lim\limits_{x\to 3}(2x^2 - 1) = 2 \times 3^2 - 1 = 17$

$\therefore \lim\limits_{x\to 3} f(x) = f(3)$

Therefore, f is continuous at $x = 3$

Q.3 Examine the following functions for continuity.

(a) $f(x) = x - 5$ (b) $f(x) = \frac{1}{x-5}, x \neq 5$ (c) $f(x) = \frac{x^2-25}{x+5}, x \neq 5$ (d) $f(x) = |x - 5|$

Sol: (a) The given function is $f(x) = x - 5$

It is evident that f is defined at every real number k and its value at k is $k - 5$.

It is also observed that, $\lim\limits_{x\to k} f(x) = \lim\limits_{x\to k}(x - 5) = k - 5 = f(k)$

$\therefore \lim\limits_{x\to k} f(x) = f(k)$

Hence, f is continuous at every real number and therefore, it is a continuous function.

(b) The given function is $f(x) = \frac{1}{x-5}, x \neq 5$

For any real number $k \neq 5$, we obtain

$$\lim_{x\to k} f(x) = \lim_{x\to k} \frac{1}{x-5} = \frac{1}{x-5}$$

Also, $f(k) = \frac{1}{x-5}$ (As $k \neq 5$)

$$\therefore \lim_{x\to k} f(x) = f(k)$$

Hence, f is continuous at every point in the domain of f and therefore, it is a continuous function.

(c) The given function is $f(x) = \frac{x^2-25}{x+5}$, $x \neq 5$

For any real number $c \neq -5$, we obtain

$$\lim_{x\to c} f(x) = \lim_{x\to c} \frac{x^2-25}{x+5} = \lim_{x\to c} \frac{(x-5)(x-5)}{x+5} = \lim_{x\to c} (x-5) = (c-5)$$

Also, $f(c)\ \frac{(c+5)(c-5)}{c+5} = (c-5)$ (as $c \neq -5$)

$$\therefore \lim_{x\to c} f(x) = f(c)$$

Hence, f is continuous at every point in the domain of f and therefore, it is a continuous function.

(d) The given function is $f(x) = |x-5| = \begin{cases} 5-x, \text{if } x < 5 \\ x-5, \text{if } x \geq 5 \end{cases}$

This function f is defined at all points of the real line.

Let c be a point on a real line. Then, $c < 5$ or $c = 5$ or $c > 5$

Case I: $c < 5$

Then, $f(c) = 5 - c$

$$\lim_{x\to c} f(x) = \lim_{x\to c}(5-x) = 5-c$$

$$\therefore \lim_{x\to c} f(x) = f(c)$$

Therefore, f is continuous at all real numbers less than 5.

Case II : $c = 5$

Then, $f(c) = f(5) = (5-5) = 0$

$$\lim_{x\to 5^+} f(x) = \lim_{x\to 5}(5-x) = (5-5) = 0$$

$$\lim_{x\to 5^+} f(x) = \lim_{x\to 5}(x-5) = 0$$

$$\therefore \lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = f(c)$$

Therefore, f is continuous at $x = 5$

Case III: $c > 5$

Then, $f(c) = f(5) = c-5$

$$\lim_{x\to c} f(x) = \lim_{x\to c}(5-x) = c-5$$

$$\therefore \lim_{x\to c^-} f(x) = f(c)$$

Therefore, f is continuous at all real numbers greater than 5.

Hence, f is continuous at every real number and therefore, it is a continuous function.

Q.4 Prove that the function $f(x) = x^n$ is continuous at $x = n$, where n is a positive integer.

Sol: The given function is $f(x) = x^n$

It is evident that f is defined at all positive integers, n, and its value at n is n^n.

$\lim_{x\to n} f(n) = \lim_{x\to n}(x^n) = n^n$

$\therefore \lim_{x\to n} f(x) = f(n)$

Therefore, f is continuous at n, where n is a positive integer.

Q.5 Is the function f defined by

$f(x) = \begin{cases} x, \text{ if } x \le 1 \\ 5, \text{ if } x > 1 \end{cases}$

continuous at $x = 0$? At $x = 1$? At $x = 2$?

Sol: The given function f is $f(x) = \begin{cases} x, \text{ if } x \le 1 \\ 5, \text{ if } x > 1 \end{cases}$

At $x = 0$,

It is evident that f is defined at 0 and its value at 0 is 0.

$\lim_{x\to 0} f(x) = \lim_{x\to 0}(x) = 0$

$\therefore \lim_{x\to 0} f(x) = f(0)$

Therefore, f is continuous at $x = 0$

At $x = 1$,

f is defined at 1 and its value at 1 is 1.

The left-hand limit of f at $x = 1$ is,

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(x) = 1$

The right-hand limit of f at $x = 1$ is,

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(5) = 5$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(x)$

Therefore, f is not continuous at $x = 1$

At $x = 2$,

f is defined at 2 and its value at 2 is 5.

$\lim_{x\to 2} f(x) = \lim_{x\to 2}(5) = 5$

$\therefore \lim_{x\to 2} f(x) = f(2)$

Therefore, f is continuous at $x = 2$

Q.6 Find all points of discontinuity of f, where f is defined by

$f(x) = \begin{cases} 2x + 3, \text{ if } x \le 2 \\ 2x - 3, \text{ if } x > 2 \end{cases}$

Sol: The given function f is $f(x) = \begin{cases} 2x + 3, \text{ if } x \le 2 \\ 2x - 3, \text{ if } x > 2 \end{cases}$

It is evident that the given function f is defined at all the points of the real line.

Let c be a point on the real line. Then, three cases arise.

(i) $c < 2$

(ii) $c > 2$

(iii) $c = 2$

Case (i) $c < 2$

Then, $f(c) = 2c + 3$

$\lim_{x \to c} f(x) = \lim_{x \to c} (2x + 3) = 2c + 3$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x < 2$

Case (ii) $c > 2$

Then, $f(c) = 2c - 3$

$\lim_{x \to c} f(x) = \lim_{x \to c} (2x - 3) = 2c - 3$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x > 2$

Case (iii) $c = 2$

Then, the left-hand limit of f at $x = 2$ is,

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (2x + 3) = 2 \times 2 + 3 = 7$

The right-hand limit of f at $x = 2$ is,

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (2x - 3) = 2 \times 2 - 3 = 1$

It is observed that the left- and right-hand limit of f at $x = 2$ do not coincide.

Therefore, f is not continuous at $x = 2$

Hence, $x = 2$ is the only point of discontinuity of f.

Q.7 Find all points of discontinuity of f, where f is defined by

$$f(x) = \begin{cases} |x| + 3, & \text{if } x \le -3 \\ -2x, & \text{if } -3 < x < 3 \\ 6x + 2, & \text{if } x \ge 3 \end{cases}$$

Sol: The given function f is $f(x) = \begin{cases} |x| + 3, & \text{if } x \le -3 \\ -2x, & \text{if } -3 < x < 3 \\ 6x + 2, & \text{if } x \ge 3 \end{cases}$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c < -3$, then $f(c) = -c + 3$

$\lim_{x \to c} f(x) = \lim_{x \to c} (-x + 3) = -c + 3$

$\therefore \lim_{x\to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x < -3$

Case II:

If $c < -3$, then $f(-3) = (-3) + 3 = 6$

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (-x + 3) = -(-3) + 3 = 7$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (-x + 3) = -2 \times (-3) = 6$

$\therefore \lim_{x\to 3} f(x) = f(-3)$

Therefore, f is continuous at $x = -3$

Case III:

If $-3 < c < 3$, then $f(c) = -c$ and $\lim_{x\to c} f(x) = \lim_{x\to c}(-2x) = -2c$

$\therefore \lim_{x\to c} f(x) = f(c)$

Therefore, f is continuous in $(-3, 3)$.

Case IV:

If $c = 3$, then the left-hand limit of f at $x = 3$ is,

limx→3−fx=limx→3−(−2x) limx→3+fx= $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (6x + 2) = -2\times 3 = -6$

The right-hand limit of f at $x = 3$ is,

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (6x + 2) = 6 \times 3 + 2 = 20$

It is observed that the left- and right-hand limit of f at $x = 3$ do not coincide.

Therefore, f is not continuous at $x = 3$

Case V:

If $c < 3$, then $f(c) = 6c + 2$ and $\lim_{x\to c} f(x) = \lim_{x\to c}(6x + 2) = 6c + 2$

$\therefore \lim_{x\to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x > 3$

Hence, $x = 3$ is the only point of discontinuity of f.

Q.8 Find all points of discontinuity of f, where f is defined by

$$f(x) = \begin{cases} \frac{|x|}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Sol: The given function f is $f(x) = \begin{cases} \frac{|x|}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$

It is known that, $x < 0 \Rightarrow |x| = -x$ and $x > 0 \Rightarrow |x| = x$

Therefore, the given function can be rewritten as

$$f(x) = \begin{cases} \frac{|x|}{x} = \frac{-x}{x} = -1, \text{ if } x < 0 \\ 0, \text{ if } x = 0 \\ \frac{|x|}{x} = \frac{x}{x} = 1, \text{ if } x > 0 \end{cases}$$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c < 0$, then $f(c) = -1$

$\lim_{x \to c} f(x) = \lim_{x \to c}(-1) = -1$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points $x < 0$

Case II:

If $c = 0$, then the left-hand limit of f at $x = 0$ is,

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-1) = -1$

The right-hand limit of f at $x = 0$ is,

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (1) = 1$

It is observed that the left- and right-hand limit of f at $x = 0$ do not coincide.

Therefore, f is not continuous at $x = 0$

Case III:

If $c < 0$, then $f(c) = 1$

$\lim_{x \to c} f(x) = \lim_{x \to c}(-1) = 1$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x > 0$

Hence, $x = 0$ is the only point of discontinuity of f.

Q.9 Find all points of discontinuity of f, where f is defined by

$$f(x) = \begin{cases} \frac{x}{|x|}, \text{ if } x < 0 \\ -1, \text{ if } x \geq 0 \end{cases}$$

Sol: The given function f is $f(x) = \begin{cases} \frac{x}{|x|}, \text{ if } x < 0 \\ -1, \text{ if } x \geq 0 \end{cases}$

It is known that, $x < 0 \Rightarrow |x| = -x$

Therefore, the given function can be rewritten as

$$f(x) = \begin{cases} \frac{x}{|x|} = \frac{x}{-x} = -1, \text{ if } x < 0 \\ -1, \text{ if } x \geq 0 \end{cases}$$

$\Rightarrow f(x) = -1$ for all $x \in R$

Let c be any real number. Then, $\lim_{x \to c} f(x) = \lim_{x \to c}(-1) = -1$

Also, $f(c) = -1 = \lim_{x \to c} f(x)$

Therefore, the given function is a continuous function.

Hence, the given function has no point of discontinuity

Q.10 Find all points of discontinuity of f, where f is defined by

$$f(x) = \begin{cases} x+1, & \text{if } x \geq 1 \\ x^2+1, & \text{if } x > 1 \end{cases}$$

Sol: The given function f is $f(x) = \begin{cases} x+1, & \text{if } x \geq 1 \\ x^2+1, & \text{if } x > 1 \end{cases}$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c < 1$, then $f(c) = c^2 + 1$ and $\lim_{x \to c} f(x) = \lim_{x \to c}(x^2+1) = c^2 + 1$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x < 1$

Case II:

If $c = 1$, then $f(c) = f(1) = 1 + 1 = 2$

The left-hand limit of f at $x = 1$ is,

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-}(x^2+1) = 1^2 + 1 = 2$

The right-hand limit of f at $x = 1$ is,

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+}(x+1) = 1 + 1 = 2$

$\therefore \lim_{x \to 1} f(x) = f(1)$

Therefore, f is continuous at $x = 1$

Case III:

If $c > 1$, then $f(c) = c + 1$

$\lim_{x \to c} f(x) = \lim_{x \to c}(x+1) = c + 1$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x > 1$

Hence, the given function f has no point of discontinuity.

Q.11 Find all points of discontinuity of f, where f is defined by

$$f(x) = \begin{cases} x^3-3, & \text{if } x \leq 2 \\ x^2+1, & \text{if } x > 2 \end{cases}$$

Sol: The given function f is $f(x) = \begin{cases} x^3-3, & \text{if } x \leq 2 \\ x^2+1, & \text{if } x > 2 \end{cases}$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c < 2$, then $f(c) = c^3 - 3$ and $\lim_{x\to c} f(x) = \lim_{x\to c}(x^3 - 3) = c^3 - 3$

$\therefore \lim_{x\to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x < 2$

Case II:

If $c = 2$, then $f(c) = f(2) = 2^3 - 3 = 5$

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x^3 - 3) = 2^3 - 3 = 5$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x^2 + 1) = 2^2 + 1 = 5$

$\therefore \lim_{x\to 2} f(x) = f(2)$

Therefore, f is continuous at $x = 2$

Case III:

If $c > 2$, then $f(c) = c^2 + 1$

$\lim_{x\to c} f(x) = \lim_{x\to c}(x^2 + 1) = c^2 + 1$

$\therefore \lim_{x\to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x > 2$

Thus, the given function f is continuous at every point on the real line.

Hence, f has no point of discontinuity.

Q.12 Find all points of discontinuity of f, where f is defined by

$$f(x) = \begin{cases} x^{10} - 1, & \text{if } x \leq 1 \\ x^2, & \text{if } x > 1 \end{cases}$$

Sol: The given function f is $f(x) = \begin{cases} x^{10} - 1, & \text{if } x \leq 1 \\ x^2, & \text{if } x > 1 \end{cases}$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c < 2$, then $f(c) = c^{10} - 1$ and $\lim_{x\to c} f(x) = \lim_{x\to c}(x^{10} - 1) = c^{10} - 1$

$\therefore \lim_{x\to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x < 1$

Case II:

If $c = 1$, then the left-hand limit of f at $x = 1$ is,

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x^{10} - 1) = 1^{10} - 1 = 1 - 1 = 0$

The right-hand limit of f at x = 1 is,

$\lim\limits_{x\to1^+} f(x) = \lim\limits_{x\to1^+}(x^2) = 1^2 = 1$

It is observed that the left- and right-hand limit of f at x = 1 do not coincide.

Therefore, f is not continuous at x = 1

Case III:

If $c > 1$, then $f(c) = c^2$

$\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c}(x^2) = c^2$

$\therefore \lim\limits_{x\to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x > 1$

Thus, from the above observation, it can be concluded that x = 1 is the only point of discontinuity of f.

Q.13 Is the function defined by $f(x) = \begin{cases} x+5, & \text{if } x \le 1 \\ x-5, & \text{if } x > 1 \end{cases}$ a continuous function?

Sol: The given function is by $f(x) = \begin{cases} x+5, & \text{if } x \le 1 \\ x-5, & \text{if } x > 1 \end{cases}$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c < 1$, then $f(c) = c + 5$ and $\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c}(x+5) = c + 5$

$\therefore \lim\limits_{x\to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x < 1$

Case II:

If $c = 1$, then $f(1) = 1 + 5 = 6$

The left-hand limit of f at x = 1 is,

limx→1−fx=limx→1−(x+5) $\lim\limits_{x\to1^-} f(x) = \lim\limits_{x\to1^-}(x+5) = 1+5 = 6$

The right-hand limit of f at x = 1 is,

$\lim\limits_{x\to1^+} f(x) = \lim\limits_{x\to1^+}(x-5) = 1 - 5 = -4$

It is observed that the left- and right-hand limit of f at x = 1 do not coincide.

Therefore, f is not continuous at x = 1

Case III:

If $c < 1$, then $f(c) = c - 5$ and $\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c}(x-5) = c - 5$

$\therefore \lim\limits_{x\to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x > 1$

Thus, from the above observation, it can be concluded that x = 1 is the only point of discontinuity of f.

Q.14 Discuss the continuity of the function f, where f is defined by

$$f(x) = \begin{cases} 3, & \text{if } 0 \le x \le 1 \\ 4, & \text{if } 1 < x < 3 \\ 5, & \text{if } 3 \le x \le 10 \end{cases}$$

Sol: The given function f is $f(x) = \begin{cases} 3, & \text{if } 0 \le x \le 1 \\ 4, & \text{if } 1 < x < 3 \\ 5, & \text{if } 3 \le x \le 10 \end{cases}$

The given function is defined at all points of the interval [0, 10].

Let c be a point in the interval [0, 10].

Case I:

If $0 \le x \le 1$, then $f(c) = 3$ and $\lim_{x \to c} f(x) = \lim_{x \to c}(3) = 3$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x < -3$

Case II:

If $c = 1$, then $f(3) = 3$

The left-hand limit of f at $x = 1$ is,

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-}(3) = 3$

The right-hand limit of f at $x = 1$ is,

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+}(4) = 4$

It is observed that the left- and right-hand limits of f at $x = 1$ do not coincide.

Therefore, f is not continuous at $x = 1$

Case III:

If $1 < c < 3$, then $f(c) = 4$ and $\lim_{x \to c} f(x) = \lim_{x \to c}(4) = 4$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points of the interval (1, 3).

Case IV:

If $c = 3$, then $f(c) = 5$

The left-hand limit of f at $x = 3$ is,

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-}(4) = 4$

The right-hand limit of f at $x = 3$ is,

$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+}(5) = 5$

It is observed that the left- and right-hand limits of f at $x = 3$ do not coincide.

Therefore, f is not continuous at $x = 3$

Case V:

If $3 < c \le 10$, then $f(c) = 5$ and $\lim_{x \to c} f(x) = \lim_{x \to c}(5) = 5$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points of the interval (3, 10].

Hence, f is not continuous at x = 1 and x = 3.

Q.15 Discuss the continuity of the function f, where f is defined by

$$f(x) = \begin{cases} 2x, & \text{if } x < 0 \\ 0, & \text{if } 0 \le x \le 1 \\ 4x, & \text{if } x > 1 \end{cases}$$

Sol: The given function f is $f(x) = \begin{cases} 2x, & \text{if } x < 0 \\ 0, & \text{if } 0 \le x \le 1 \\ 4x, & \text{if } x > 1 \end{cases}$

The given function is defined at all points of the real line.

Let c be a point on the real line.

Case I:

If $c < 0$, then $f(c) = 2c$

$\lim_{x \to c} f(x) = \lim_{x \to c} (2x) = 2c$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that x < 0

Case II:

If $c = 0$, then $f(c) = f(0) = 0$

The left-hand limit of f at x = 0 is,

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (2x) = 2 \times 0 = 0$

The right-hand limit of f at x = 0 is,

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (0) = 0$

$\therefore \lim_{x \to c} f(x) = f(0)$

Therefore, f is not continuous at x = 0

Case III:

If $0 < c < 1$, then $f(x) = 0$ and $\lim_{x \to c} f(x) = \lim_{x \to c} (0) = 0$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points of the interval (0,1).

Case IV:

If $c = 1$, then $f(c) = f(1) = 0$

The left-hand limit of f at x = 1 is,

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (0) = 0$

The right-hand limit of f at x = 1 is,

$\lim_{x\to1^+} f(x) = \lim_{x\to1^+} (4x) = 4 \times 1 = 4$

It is observed that the left- and right-hand limits of f at x = 1 do not coincide.

Therefore, f is not continuous at x = 1

Case V:

If $c < 1$, then $f(c) = 4c$ and $\lim_{x\to c} f(x) = \lim_{x\to c}(4x) = 4c$

$\therefore \lim_{x\to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that x > 1

Hence, f is not continuous only at x = 1

Q.16 Discuss the continuity of the function f, where f is defined by

$$f(x) = \begin{cases} -2, & \text{if } x \le -1 \\ 2x, & \text{if } -1 \le x \le 1 \\ 2, & \text{if } x > 1 \end{cases}$$

Sol: The given function f is $f(x) = \begin{cases} -2, & \text{if } x \le -1 \\ 2x, & \text{if } -1 \le x \le 1 \\ 2, & \text{if } x > 1 \end{cases}$

The given function is defined at all points of the real line.

Let c be a point on the real line.

Case I:

If $c < -1$, then $f(c) = -2$ and $\lim_{x\to c} f(x) = \lim_{x\to c}(-2) = -2$

$\therefore \lim_{x\to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that x < −1

Case II:

If $c = -1$, then $f(c) = f(-1) = -2$

The left-hand limit of f at x = −1 is,

$\lim_{x\to1^-} f(x) = \lim_{x\to1^-} (-2) = -2$

The right-hand limit of f at x = −1 is,

$\lim_{x\to1^+} f(x) = \lim_{x\to1^+} (2x) = 2 \times (-1) = -2$

$\therefore \lim_{x\to1} f(x) = f(-1)$

Therefore, f is continuous at x = −1

Case III:

If $-1 < c < 1$, then $f(c) = 2c$

$\lim_{x\to c} f(x) = \lim_{x\to c}(2x) = 2c$

$\therefore \lim_{x\to c} f(x) = f(c)$

Therefore, f is continuous at all points of the interval (−1, 1).

Case IV:

If $c = 1$, then $f(c) = f(1) = 2 \times 1 = 2$

The left-hand limit of f at x = 1 is,

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (-2) = 2 \times 1 = 2$

The right-hand limit of f at x = 1 is,

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (2) = 2$

$\therefore \lim_{x \to 1} f(x) = f(c)$

Therefore, f is continuous at x = 2

Case V:

If $c > 1$, then $f(c) = 2$ and $\lim_{x \to c} f(x) = \lim_{x \to c} (2) = 2$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x > 1$

Thus, from the above observations, it can be concluded that f is continuous at all points of the real line.

Q.17 Find the relationship between a and b so that the function f defined by

$f(x) = \begin{cases} ax + 1, & \text{if } x \le 3 \\ bx + 3, & \text{if } x > 3 \end{cases}$ is continuous at x = 3.

Sol: The given function f is $f(x) = \begin{cases} ax + 1, & \text{if } x \le 3 \\ bx + 3, & \text{if } x > 3 \end{cases}$

If f is continuous at x = 3, then

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} f(x) = f(3)$ …….. (1)

Also,

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} f(ax + 1) = 3a + 1$

$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} f(bx + 3) = 3b + 3$

$f(3) = 3a + 1$

Therefore, from (1), we obtain

$3a + 1 = 3b + 3 = 3a + 1$

$\Rightarrow 3a + 1 = 3b + 3$

$\Rightarrow 3a = 3b + 2$

$\Rightarrow a = b + \frac{2}{3}$

Therefore, the required relationship is given by, $a = b + \frac{2}{3}$

Q.18 For what value of is the function defined by

$f(x) = \begin{cases} \lambda(x^2 - 2x), & \text{if } x \le 0 \\ 4x + 1, & \text{if } x > 0 \end{cases}$ continuous at x = 0? What about continuity at x = 1?

Sol: The given function f is $f(x) = \begin{cases} \lambda(x^2 - 2x), & \text{if } x \le 0 \\ 4x + 1, & \text{if } x > 0 \end{cases}$

If f is continuous at x =0, then

$\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} f(x) = f(0)$

$\Rightarrow \lim\limits_{x \to 0^-} \lambda(x^2 - 2x) = \lim\limits_{x \to 0^-} (4x + 1) = \lambda(0^2 - 2 \times 0)$

$\Rightarrow \lambda(0^2 - 2 \times 0) = 4 \times 0 + 1 = 0$

$\Rightarrow 0 = 1 = 0$, Which is not possible

Therefore, there is no value of λ for which f is continuous at $x = 0$

At $x = 1$,

$f(1) = 4x + 1 = 4 \times 1 + 1 = 5$

$\lim\limits_{x \to 1} f(4x + 1) = 4 \times 1 + 1 = 5$

$\therefore \lim\limits_{x \to 1} f(x) = f(1)$

Therefore, for any values of λ, f is continuous at $x = 1$

Q.19 Show that the function defined by $g(x) = x - [x]$ is discontinuous at all integral point. Here [x] denotes the greatest integer less than or equal to x.

Sol: The given function is $g(x) = x - [x]$

It is evident that g is defined at all integral points.

Let n be an integer.

Then,

$g(n) = n - [n] = n - n = 0$

The left-hand limit of f at $x = n$ is,

$\lim\limits_{x \to n^-} g(x) = \lim\limits_{x \to n^-} (x - [x]) = \lim\limits_{x \to n^-} (x) = \lim\limits_{x \to n^-} ([x]) = n - (n - 1) = 1$

The right-hand limit of f at $x = n$ is,

$\lim\limits_{x \to n^+} g(x) = \lim\limits_{x \to n^+} (x - [x]) = \lim\limits_{x \to n^+} (x) = \lim\limits_{x \to n^+} ([x]) = n - n = 0$

It is observed that the left- and right-hand limits of f at $x = n$ do not coincide.

Therefore, f is not continuous at $x = n$

Hence, g is discontinuous at all integral points.

Q.20 Is the function defined by $f(x) = x^2 - \sin x + 5$ continuous at $x = p$?

Sol: The given function is $f(x) = x^2 - \sin x + 5$

It is evident that f is defined at $x = p$

At $x = \pi, f(x) = f(\pi) = \pi^2 - \sin\pi + 5 = \pi^2 - 0 + 5 = \pi^2 + 5$

Consider $\lim\limits_{x \to \pi} f(x) = \lim\limits_{x \to \pi} f(x^2 - \sin x + 5)$

Put $x = \pi + h$

If $x \to \pi$, then it is evident that $h \to 0$

$$\lim_{x\to\pi} f(x) = \lim_{x\to\pi} f(x^2 - \sin x + 5)$$

$$= \lim_{h\to 0}[(\pi + h)^2 - \sin(\pi + h) + 5]$$

$$= \lim_{h\to 0}(\pi + h)^2 - \lim_{h\to 0}\sin(\pi + h) + \lim_{h\to 0}5$$

$$= (\pi + 0)^2 - \lim_{h\to 0}[\sin\pi\cos h + \cos\pi\sin h] + 5$$

$$= \pi^2 - \lim_{h\to 0}\sin\pi\cos h - \lim_{h\to 0}\cos\pi\sin h + 5$$

$$= \pi^2 - \sin\pi\cos 0 - \cos\pi\sin 0 + 5$$

$$= \pi^2 - 0 \times 1 - (-1) \times 0 + 5$$

$$= \pi^2 + 5$$

$$\therefore \lim_{x\to\pi} f(x) = f(\pi)$$

Therefore, the given function f is continuous at $x = \pi$

Q.21 Discuss the continuity of the following functions.

(a) $f(x) = \sin x + \cos x$

(b) $f(x) = \sin x - \cos x$

(c) $f(x) = \sin x \times \cos x$

Sol: It is known that if g and h are two continuous functions, then $g + h$, $g - h$, and $g.h$ are also continuous.

It has to prove first that $g(x) = \sin x$ and $h(x) = \cos x$ are continuous functions.

Let $g(x) = \sin x$

It is evident that $g(x) = \sin x$ is defined for every real number.

Let c be a real number. Put $x = c + h$

If $x \to c$, then $h \to 0$

$g(c) = \sin c$

$$\lim_{x\to c} g(x) = \lim_{x\to c}\sin x$$

$$= \lim_{h\to 0}\sin(c + h)$$

$$= \lim_{h\to 0}[\sin c\cos h + \cos c\sin h]$$

$$= \lim_{h\to 0}[(\sin c\cos h) + \lim_{h\to 0}(\cos c\sin h)]$$

$$= \sin c\cos 0 + \cos c\sin 0$$

$$= \sin c + 0$$

$$= \sin c$$

$$\therefore \lim_{x\to c} g(x) = g(c)$$

Therefore, g is a continuous function.

Let $h(x) = \cos x$

It is evident that $h(x) = \cos x$ is defined for every real number.

Let c be a real number. Put $x = c + h$

If $x \to c$, then $h \to 0$

$h(c) = \cos c$

$\lim_{x \to c} h(x) = \lim_{x \to c} \cos x$

$= \lim_{h \to 0} \cos(c + h)$

$= \lim_{h \to 0} [\cos c \cos h - \sin c \sin h]$

$= \lim_{h \to 0} (\cos c \cos h) + \lim_{h \to 0} (\sin c \sin h)$

$= \cos c \cos 0 + \sin c \sin 0$

$= \cos c \times 1 + \sin c \times 0$

$= \cos c$

$\therefore \lim_{x \to c} h(x) = h(c)$

Therefore, h is a continuous function.

Therefore, it can be concluded that

(a) $f(x) = g(x) + h(x) = \sin x + \cos x$ is a continuous function

(b) $f(x) = g(x) - h(x) = \sin x - \cos x$ is a continuous function

(c) $f(x) = g(x) \times h(x) = \sin x \times \cos x$ is a continuous function

Q.22 Discuss the continuity of the cosine, cosecant, secant and cotangent functions.

Sol: It is known that if g and h are two continuous functions, then

(i) $\frac{h(x)}{g(x)}$, $g(x) \neq 0$ is continuous

(i) $\frac{1}{g(x)}$, $g(x) \neq 0$ is continuous

(i) $\frac{1}{h(x)}$, $h(x) \neq 0$ is continuous

It has to be proved first that $g(x) = \sin x$ and $h(x) = \cos x$ are continuous functions.

Let $g(x) = \sin x$

It is evident that $g(x) = \sin x$ is defined for every real number.

Let c be a real number. Put $x = c + h$

If $x \to c$, then $h \to 0$

$g(c) = \sin c$

$\lim_{x \to c} g(x) = \lim_{x \to c} \sin x$

$= \lim_{h \to 0} \sin(c + h)$

$= \lim_{h \to 0} [\sin c \cos h + \cos c \sin h]$

$= \lim_{h \to 0} [(\sin c \cos h) + \lim_{h \to 0} (\cos c \sin h)]$

$= \sin c \cos 0 + \cos c \sin 0$

$= \sin c + 0$

$= \sin c$

$\therefore \lim_{x \to c} g(x) = g(c)$

Therefore, g is a continuous function.

Let h (x) = cos x

It is evident that h (x) = cos x is defined for every real number.

Let c be a real number. Put x = c + h

If x ® c, then h ® 0

h (c) = cos c

h (c) = cos c

$\lim_{x \to c} h(x) = \lim_{x \to c} \cos x$

$= \lim_{h \to 0} \cos(c + h)$

$= \lim_{h \to 0} [\cos c \cos h - \sin c \sin h]$

$= \lim_{h \to 0} (\cos c \cos h) + \lim_{h \to 0} (\sin c \sin h)$

$= \cos c \cos 0 + \sin c \sin 0$

$= \cos c \times 1 + \sin c \times 0$

$= \cos c$

$\therefore \lim_{x \to c} h(x) = h(c)$

Therefore, h (x) = cos x is continuous function.

It can be concluded that,

$\operatorname{cosec} x = \frac{1}{\sin x}, \sin x \neq 0$ is continuous

$\Rightarrow \operatorname{cosec} x, \; x \neq n\pi (n \in z)$ is continuous

Therefore, cosecant is continuous except at x = np, n Î Z

$\sec x = \frac{1}{\cos x}, \cos x \neq 0$ is continuous

$\Rightarrow \sec x, \; x \neq (2n + 1)\frac{\pi}{2} \; (n \in z)$ is continuous

Therefore, secant is continuous except at $x = (2n + 1)\frac{\pi}{2} \; (n \in z)$

$\cot x = \frac{\cos x}{\sin x}, \; \sin x \neq 0$ is continuous

$\Rightarrow \cot x, \; x \neq n\pi \; (n \in z)$ is continuous

Therefore, cotangent is continuous except at x = np, n Î Z

Q.23 Find the points of discontinuity of f, where $f(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } x < 0 \\ x + 1, & \text{if } x \geq 0 \end{cases}$ a continuous function?

Sol: The given function is by $f(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } x < 0 \\ x+1, & \text{if } x \geq 0 \end{cases}$

It is evident that f is defined at all points of the real line.

Let c be a real number.

Case I:

If $c < 0$, then $f(c) = \frac{\sin x}{x}$ and $\lim_{x \to c} f(x) = \lim_{x \to c}\left(\frac{\sin x}{x}\right) = \frac{\sin c}{c}$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x < 0$

Case II:

If $c > 0$, then $f(c) = c + 1$ and $\lim_{x \to c} f(x) = \lim_{x \to c}(x+1) = c + 1$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x > 0$

Case III:

If $c = 0$, then $f(c) = f(0) = 0 + 1 = 1$

The left-hand limit of f at $x = 0$ is,

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-}(x+1) = 1$

The right-hand limit of f at $x = 0$ is,

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+}(x+1) = 1$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-}(x+1) = f(0)$

Therefore, f is continuous at $x = 0$

From the above observations, it can be concluded that f is continuous at all points of the real line.

Thus, f has no point of discontinuity.

Q.24 Determine if f defined by $f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x \geq 0 \end{cases}$ a continuous function?

Sol: The given function is by $f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x \geq 0 \end{cases}$

It is evident that f is defined at all points of the real line.

Let c be a real number.

Case I:

If $c \neq 0$, then $f(c) = c^2 \sin\frac{1}{c}$

$\lim_{x \to c} f(x) = \lim_{x \to c}\left(x^2 \sin\frac{1}{x}\right) = \left(\lim_{x \to c} x^2\right)\left(\lim_{x \to c} x^2 \sin\frac{1}{x}\right) = c^2 \sin\frac{1}{c}$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x < 0$

Case II:

If $c = 0$, then $f(0) = 0$

$\lim_{x\to0^-} f(x) = \lim_{x\to0^-}\left(x^2 \sin\frac{1}{x}\right) = \lim_{x\to0}\left(x^2 \sin\frac{1}{x}\right)$

It is known that, $-1 \le \sin\frac{1}{x} \le 1, \quad x \ne 0$

$\Rightarrow -x^2 \le \sin\frac{1}{x} \le x^2$

$\Rightarrow \lim_{x\to0}(-x^2) \le \lim_{x\to0}\left(x^2 \sin\frac{1}{x}\right) \le \lim_{x\to0}(x^2)$

$\Rightarrow \lim_{x\to0}\left(x^2 \sin\frac{1}{x}\right) = 0$

$\therefore \lim_{x\to0^-} f(x) = 0$

Similarly, $\lim_{x\to0^-} f(x) = \lim_{x\to0^+}\left(x^2 \sin\frac{1}{x}\right) = \lim_{x\to0^+}\left(x^2 \sin\frac{1}{x}\right) = 0$

$\therefore \lim_{x\to0^-} f(x) = f(0) = \lim_{x\to0^+} f(x)$

Therefore, f is continuous at $x = 0$

From the above observations, it can be concluded that f is continuous at every point of the real line.

Thus, f is a continuous function.

Q.25 Determine if f defined by $f(x) = \begin{cases} \sin x - \cos x, & \text{if } x \ne 0 \\ -1, & \text{if } x = 0 \end{cases}$ a continuous function?

Sol: The given function is by $f(x) = \begin{cases} \sin x - \cos x, & \text{if } x \ne 0 \\ -1, & \text{if } x = 0 \end{cases}$

It is evident that f is defined at all points of the real line.

Let c be a real number.

Case I:

If $c \ne 0$, then $f(c) = \sin x - \cos x$

$\lim_{x\to c} f(x) = \lim_{x\to c}(\sin x - \cos x) = \sin c - \cos c$

$\therefore \lim_{x\to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x < 0$

Case II:

If $c = 0$, then $f(0) = -1$

$\lim_{x\to0^-} f(x) = \lim_{x\to0^-}(\sin x - \cos x) = \sin 0 - \cos 0 = 0 - 1 = -1$

$\lim_{x\to0^+} f(x) = \lim_{x\to0^+}(\sin x - \cos x) = \sin 0 - \cos 0 = 0 - 1 = -1$

$\therefore \lim_{x\to0^-} f(x) = \lim_{x\to0^+} f(x) = f(0)$

Therefore, f is continuous at $x = 0$

From the above observations, it can be concluded that f is continuous at every point of the real line.

Thus, f is a continuous function.

Q.26 Find the values of k so that the function f is continuous at the indicated point.

$$f(x) = \begin{cases} \frac{k\cos x}{\pi - 2x}, & \text{if } x \neq \frac{\pi}{2} \\ 3, & \text{if } x = \frac{\pi}{2} \end{cases} \quad \text{at } x = \frac{\pi}{2}$$

Sol: The given function f is $f(x) = \begin{cases} \frac{k\cos x}{\pi - 2x}, & \text{if } x \neq \frac{\pi}{2} \\ 3, & \text{if } x = \frac{\pi}{2} \end{cases}$

The given function f is continuous at $x = \frac{\pi}{2}$, if f is defined at $x = \frac{\pi}{2}$ and if the value of the f at $x = \frac{\pi}{2}$ equals the limit of f at .

It is evident that f is defined at $x = \frac{\pi}{2}$ and $f\left(\frac{\pi}{2}\right) = 3$

$$\lim_{x \to \frac{\pi}{2}} f(x) = \lim_{x \to \frac{\pi}{2}} \frac{k\cos x}{\pi - 2x}$$

Put $x = x \to \frac{\pi}{2} + h$

Then, $x \to \frac{\pi}{2} + h \to 0$

$$\therefore \lim_{x \to \frac{\pi}{2}} f(x) = \lim_{x \to \frac{\pi}{2}} \frac{k\cos x}{\pi - 2x} = \lim_{h \to 0} \frac{k\cos\left(\frac{\pi}{2} + h\right)}{\pi - 2\left(\frac{\pi}{2} + h\right)} = k \lim_{h \to 0} \frac{-\sinh}{-2h} = \frac{k}{2} \lim_{h \to 0} \frac{\sinh}{h} == \frac{k}{2}.1 == \frac{k}{2}$$

$$\therefore \lim_{x \to \frac{\pi}{2}} f(x) = f\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \frac{k}{2} = 3$$

$$\Rightarrow k = 6$$

Therefore, the required value of k is 6.

Q.27 Find the values of k so that the function f is continuous at the indicated point.

$$f(x) = \begin{cases} kx^2, & \text{if } x \leq 2 \\ 3, & \text{if } x > 2 \end{cases} \quad \text{at } x = 2$$

Sol: The given function f is $f(x) = \begin{cases} kx^2, & \text{if } x \leq 2 \\ 3, & \text{if } x > 2 \end{cases}$

The given function f is continuous at x =2, if f is defined at x = 2 and if the value of the f at x = 2 equals the limit of f at x = 2.

It is evident that f is defined at x = 2 and $f(2) = k(2)^2 = 4k$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} f(x) = f(2)$$

$$\Rightarrow \lim_{x \to 2^-} (kx^2) = \lim_{x \to 2^-} (3) = 4k$$

$$\Rightarrow k \times 2^2 = 3 = 4k$$

$$\Rightarrow 4k = 3 = 4k$$

$$\Rightarrow 4k = 3$$

$\Rightarrow k = \frac{3}{4}$

Therefore, the required value of k is $\frac{3}{4}$.

Q.28 Find the values of k so that the function f is continuous at the indicated point.

$f(x) = \begin{cases} kx + 1 \ , \ \text{if } x \le \pi \\ \cos x \quad , \ \text{if } x > \pi \end{cases}$ at $x = \pi$

Sol: The given function f is $f(x) = \begin{cases} kx + 1 \ , \ \text{if } x \le \pi \\ \cos x \quad , \ \text{if } x > \pi \end{cases}$

The given function f is continuous at x =p, if f is defined at x = p and if the value of the f at x = p equals the limit of f at x = p.

It is evident that f is defined at x = p and $f(\pi) = k\pi + 1$

$\lim_{x \to \pi^-} f(x) = \lim_{x \to \pi^+} f(x) = f(\pi)$

$\Rightarrow \lim_{x \to \pi^-} (kx + 1) = \lim_{x \to \pi^+} \cos x = k\pi + 1$

$\Rightarrow k\pi + 1 = \cos\pi = k\pi + 1$

$\Rightarrow k\pi + 1 = -1 = k\pi + 1$

$\Rightarrow k = -\frac{2}{\pi}$

Therefore, the required value of k is $-\frac{2}{\pi}$

Q.29 Find the values of k so that the function f is continuous at the indicated point.

$f(x) = \begin{cases} kx + 1 \ , \ \text{if } x \le 5 \\ \cos x \quad , \ \text{if } x > 5 \end{cases}$ at $x = 5$

Sol: The given function f is $f(x) = \begin{cases} kx + 1 \ , \ \text{if } x \le 5 \\ \cos x \quad , \ \text{if } x > 5 \end{cases}$

The given function f is continuous at x =5, if f is defined at x = 5 and if the value of the f at x = 5 equals the limit of f at x = 5.

It is evident that f is defined at x = 5 and $f(5) = 5k + 1$

$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^+} f(x) = f(5)$

$\Rightarrow \lim_{x \to 5^-} (kx + 1) = \lim_{x \to 5^+} (3x - 5) = 5k + 1$

$\Rightarrow 5k + 1 = 15 - 5 = 5k + 1$

$\Rightarrow 5k + 1 = 10$

$\Rightarrow k = \frac{9}{5}$

Therefore, the required value of k is $\frac{9}{5}$

Q.30 Find the values of a and b such that the function defined by

$f(x) = \begin{cases} 5 \quad\quad , \ \text{if } x \le 2 \\ ax + b, \ \text{if } 2 < x < 10 \\ 21 \quad\quad , \ \text{if } x \ge 10 \end{cases}$ is a continuous function.

Sol: The given function f is $f(x) = \begin{cases} 5 & , \text{ if } x \leq 2 \\ ax + b, & \text{if } 2 < x < 10 \\ 21 & , \text{ if } x \geq 10 \end{cases}$

It is evident that the given function f is defined at all points of the real line.

If f is a continuous function, then f is continuous at all real numbers.

In particular, f is continuous at x = 2 and x = 10

Since f is continuous at x = 2, we obtain

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2)$

$\Rightarrow \lim_{x \to 2^-} (5) = \lim_{x \to 2^+} (ax + b) = 5$

$\Rightarrow 5 = 2a + b = 5$

$\Rightarrow 2a + b = 5$ ...... (1)

Since f is continuous at x = 10, we obtain

$\lim_{x \to 10^-} f(x) = \lim_{x \to 10^+} f(x) = f(10)$

$\Rightarrow \lim_{x \to 10^-} (ax + b) = \lim_{x \to 10^+} (21) = 21$

$\Rightarrow 10a + b = 21 = 21$

$\Rightarrow 10a + b = 21$ ...... (2)

On subtracting equation (1) from equation (2), we obtain

$8a = 16$

$\Rightarrow a = 2$

By putting a = 2 in equation (1), we obtain

$2 \times 2 + b = 5$

$\Rightarrow 4 + b = 5$

$\Rightarrow b = 1$

Therefore, the values of a and b for which f is a continuous function are 2 and 1 respectively.

Q.31 Show that the function defined by $f(x) = \cos(x^2)$ is a continuous function.

Sol: The given function f is $f(x) = \cos(x^2)$

This function f is defined for every real number and f can be written as the composition of two functions as,

$f = g \circ h$, where $g(x) = \cos x$ and $h(x) = x^2$

$[\because (goh)(x) = g(h(x)) = g(x^2) = \cos(x^2) = f(x)]$

It has to be first proved that $g(x) = \cos x$ and $h(x) = x^2$ are continuous functions.

It is evident that g is defined for every real number.

Let c be a real number.

Then, $g(c) = \cos c$

Put $c = c + h$

If $x \to c$, then $h \to 0$

$\lim_{x\to c} g(x) = \lim_{x\to c} \cos x$

$= \lim_{h\to 0} \cos(c+h)$

$= \lim_{h\to 0}[\cos c \cos h - \sin c \sin h]$

$= \lim_{h\to 0} \cos c \cos h - \lim_{h\to 0} \sin c \sin h$

$= \cos c \cos 0 - \sin c \sin 0$

$= \cos c \times 1 - \sin c \times 0$

$= \cos c$

$\therefore \lim_{x\to c} g(x) = g(c)$

Therefore, g (x) = cos x is continuous function. $h(x) = x^2$

Clearly, h is defined for every real number.

Let k be a real number, then $h(k) = k^2$

$\lim_{x\to k} h(x) = \lim_{x\to k} x^2 = k^2$

$\therefore \lim_{x\to k} h(x) = h(k)$

Therefore, h is a continuous function.

It is known that for real valued functions g and h, such that (g o h) is defined at c, if g is continuous at c and if f is continuous at g (c), then (f o g) is continuous at c.

Therefore, $f(x) = (goh)(x) = \cos(x^2)$ is a continuous function.

Q.32 Show that the function defined by $f(x) = |\cos x|$ is a continuous function.

Sol: The given function f is $f(x) = |\cos x|$

This function f is defined for every real number and f can be written as the composition of two functions as, f = g o h, where $g(x) = |x|$ and $h(x) = \cos x$

$[\because (goh)(x) = g(h(x)) = g(\cos x) = |\cos x| = f(x)]$

It has to be first proved that $g(x) = |x|$ and $h(x) = \cos x$ are continuous functions.

$g(x) = |x|$ can be written as

$$g(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

Clearly, g is defined for all real numbers.

Let c be a real number.

Case I:

If $c > 0$, Then, $g(c) = -c$ and $\lim_{x\to c} g(x) = \lim_{x\to c}(-x) = -c$

$\therefore \lim_{x\to c} g(x) = g(c)$

Therefore, g is continuous at all points x, such that $x < 0$

Case II:

If $c > 0$, Then, $g(c) = c$ and $\lim_{x \to c} g(x) = \lim_{x \to c} (x) = c$

$\therefore \lim_{x \to c} g(x) = g(c)$

Therefore, g is continuous at all points x, such that $x > 0$

Case III:

If $c > 0$, Then, $g(c) = g(0) = 0$

$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} (-x) = 0$

$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} (x) = 0$

$\therefore \lim_{x \to 0^-} g(x) = \lim_{x \to 0^+} (x) = g(0)$

Therefore, g is continuous at $x = 0$

From the above three observations, it can be concluded that g is continuous at all points.

$h(x) = \cos x$

It is evident that $h(x) = \cos x$ is defined for every real number.

Let c be a real number. Put $x = c + h$

If $x \to c$, then $h \to 0$

$h(c) = \cos c$

$$\lim_{x \to c} h(x) = \lim_{x \to c} \cos x$$

$$= \lim_{h \to 0} \cos(c + h)$$

$$= \lim_{h \to 0} [\cos c \cos h - \sin c \sin h]$$

$$= \lim_{h \to 0} \cos c \cos h - \lim_{h \to 0} \sin c \sin h$$

$$= \cos c \cos 0 - \sin c \sin 0$$

$$= \cos c \times 1 - \sin c \times 0$$

$$= \cos c$$

$\therefore \lim_{x \to c} h(x) = h(c)$

Therefore, $h(x) = \cos x$ is a continuous function.

It is known that for real valued functions g and h, such that (g o h) is defined at c, if g is continuous at c and if f is continuous at g (c), then (f o g) is continuous at c.

Therefore, $f(x) = (goh)(x) = g(h(x)) = g(\cos x) = |\cos x|$ is a continuous function.

Q.33 Show that the function defined by $f(x) = \sin|x|$ is a continuous function.

Sol: The given function f is $f(x) = \sin|x|$

This function f is defined for every real number and f can be written as the composition of two functions as, $f = g \text{ o } h$, where $g(x) = |x|$ and $h(x) = \sin x$

$[\because (goh)(x) = g(h(x)) = g(\sin x) = |\sin x| = f(x)]$

It has to be first proved that $g(x) = |x|$ and $h(x) = \sin x$ are continuous functions.

g (x) = |x| can be written as

$$g(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

Clearly, g is defined for all real numbers.

Let c be a real number.

Case I:

If $c < 0$, Then, $g(c) = -c$ and $\lim_{x\to c} g(x) = \lim_{x\to c}(-x) = -c$

$\therefore \lim_{x\to c} g(x) = g(c)$

Therefore, g is continuous at all points x, such that $x < 0$

Case II:

If $c > 0$, Then, $g(c) = c$ and $\lim_{x\to c} g(x) = \lim_{x\to c}(x) = c$

$\therefore \lim_{x\to c} g(x) = g(c)$

Therefore, g is continuous at all points x, such that $x > 0$

Case III:

If $c > 0$, Then, $g(c) = g(0) = 0$

$\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} (-x) = 0$

$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} (x) = 0$

$\therefore \lim_{x\to 0^-} g(x) = \lim_{x\to 0^+} (x) = g(0)$

Therefore, g is continuous at $x = 0$

From the above three observations, it can be concluded that g is continuous at all points.

$h(x) = \sin x$

It is evident that $h(x) = \cos x$ is defined for every real number.

Let c be a real number. Put $x = c + k$

If $x \to c$, then $k \to 0$

$h(c) = \sin c$

$$\begin{aligned} \lim_{x\to c} h(x) &= \lim_{x\to c} \sin x \\ &= \lim_{k\to 0} \sin(c+k) \\ &= \lim_{k\to 0}[\sin c \cos k + \cos c \ \sin k] \\ &= \lim_{k\to 0} \sin c \cos k + \lim_{k\to 0} \cos c \sin k \\ &= \sin c \cos 0 + \cos c \ \sin 0 \\ &= \sin c + 0 \\ &= \sin c \end{aligned}$$

$\therefore \lim_{x\to c} h(x) = g(c)$

Therefore, h is a continuous function.

It is known that for real valued functions g and h, such that (g o h) is defined at c, if g is continuous at c and if f is continuous at g (c), then (f o g) is continuous at c.

Therefore, $f(x) = (goh)(x) = g\big(h(x)\big) = g(\sin x) = |\sin x|$ is a continuous function.

Q.34 Find all the points of discontinuity of f defined by $f(x) = |x| - |x+1|$.

Sol: The given function f is $f(x) = |x| - |x+1|$

The two functions, g and h, are defined as

$g(x) = |x|$ and $h(x) = |x+1|$

Then, $f = g - h$

The continuity of g and h is examined first.

$g(x) = |x|$ can be written as

$$g(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

Clearly, g is defined for all real numbers.

Let c be a real number.

Case I:

If $c < 0$, Then, $g(c) = -c$ and $\lim\limits_{x\to c} g(x) = \lim\limits_{x\to c}(-x) = -c$

$\therefore \lim\limits_{x\to c} g(x) = g(c)$

Therefore, g is continuous at all points x, such that $x < 0$

Case II:

If $c > 0$, Then, $g(c) = c$ and $\lim\limits_{x\to c} g(x) = \lim\limits_{x\to c}(x) = c$

$\therefore \lim\limits_{x\to c} g(x) = g(c)$

Therefore, g is continuous at all points x, such that $x > 0$

Case III:

If $c > 0$, Then, $g(c) = g(0) = 0$

$\lim\limits_{x\to 0^-} g(x) = \lim\limits_{x\to 0^-}(-x) = 0$

$\lim\limits_{x\to 0^+} g(x) = \lim\limits_{x\to 0^+}(x) = 0$

$\therefore \lim\limits_{x\to 0^-} g(x) = \lim\limits_{x\to 0^+}(x) = g(0)$

Therefore, g is continuous at $x = 0$

From the above three observations, it can be concluded that g is continuous at all points.

$h(x) = |x+1|$ can be written as

$$h(x) = \begin{cases} -(x+1), & \text{if } x < -1 \\ x+1, & \text{if } x \geq -1 \end{cases}$$

Clearly, h is defined for every real number.

Let c be a real number.

Case I:

If $c < -1$, Then, $h(c) = -(c+1)$ and $\lim_{x\to c} h(x) = \lim_{x\to c}[-(x+1)] = -(c+1)$

$\therefore \lim_{x\to c} h(x) = h(c)$

Therefore, h is continuous at all points x, such that $x < -1$

Case II:

If $c < -1$, Then, $h(c) = -(c+1)$ and $\lim_{x\to c} h(x) = \lim_{x\to c}[(x+1)] = (c+1)$

$\therefore \lim_{x\to c} h(x) = h(c)$

Therefore, h is continuous at all points x, such that $x > -1$

Case III:

If $c = -1$, then $f(c) = h(-1) = -1 + 1 = 0$

$\lim_{x\to 1^-} h(x) = \lim_{x\to 1^-}(-(x+1)) = -(-1+1) = 0$

$\lim_{x\to 1^+} h(x) = \lim_{x\to 1^+}(x+1) = -1 + 1 = 0$

$\therefore \lim_{x\to 1^-} h(x) = \lim_{x\to 1^+} h(x) = h(-1)$

Therefore, h is continuous at $x = -1$

From the above three observations, it can be concluded that h is continuous at all points of the real line.

g and h are continuous functions. Therefore, $f = g - h$ is also a continuous function.

Therefore, f has no point of discontinuity.