SpeedLabs

MATHS

CBSE 12th

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Q.1 Verify Rolle's Theorem for the function

$f(x) = x^2 + 2x - 8, x \in [-4, 2]$

Sol: The given function, $f(x) = x^2 + 2x - 8$, being a polynomial function, is continuous in $[-4, 2]$ and is differentiable in $(-4, 2)$.

Then,

$f(-4) = (-4)^2 + 2 \times (-4) - 8 = 16 - 8 - 8 = 0$

$f(2) = (2)^2 + 2 \times 2 - 8 = 4 + 4 - 8 = 0$

$\therefore f(-4) = f(2) = 0$

$\Rightarrow$ The value of f (x) at -4 and 2 coincides.

Rolle's Theorem states that there is a point $c \in (-4, 2)$ such that $f'(c) = 0$

$f(x) = x^2 + 2x - 8$

$\Rightarrow f'(c) = 0$

$\Rightarrow 2c + 2 = 0$

$\Rightarrow c = -1$, where $c = -1 \in (-4, 2)$

Hence, Rolle's Theorem is verified for the given function.

Q.2 Examine if Rolle's Theorem is applicable to any of the following functions. Can you say something about the converse of Rolle's Theorem from these examples?

(i) $f(x) = [x]$ for $x \in [5, 9]$

(ii) $f(x) = [x]$ for $x \in [-2, 2]$

(iii) $f(x) = x^2 - 1$ for $x \in [1, 2]$

Sol: By Rolle's Theorem, for a function $f: [a, b] \to R$, if

(a) f is continuous on [a, b]

(b) f is differentiable on (a, b)

(c) f (a) = f (b)

then, there exists some $c \in (a, b)$ such that $f'(c) = 0$

Therefore, Rolle's Theorem is not applicable to those functions that do not satisfy any of the three conditions of the hypothesis.

(i) $f(x) = [x]$ for $x \in [5, 9]$

It is evident that the given function f (x) is not continuous at every integral point.

In particular, f(x) is not continuous at x = 5 and x = 9

$\Rightarrow$ f (x) is not continuous in [5, 9]

Also, $f(x) = [5] = 5$ and $f(9) = [9] = 9$

$\therefore f(5) \neq f(9)$

The differentiability of f in (5, 9) is checked as follows.

Let n be an integer such that $n \in (5, 9)$

The left-hand limit of f at $x = n$ is,

$$\lim_{h\to 0^-} \frac{f(n+h)-f(n)}{h} = \lim_{h\to 0^-} \frac{(n+h)-(n)}{h} = \lim_{h\to 0^-} \frac{n-1-n}{h} == \lim_{h\to 0^-} \frac{-1}{h} = \infty$$

The right-hand limit of f at $x = n$ is,

$$\lim_{h\to 0^+} \frac{f(n+h)-f(n)}{h} = \lim_{h\to 0^+} \frac{(n+h)-(n)}{h} = \lim_{h\to 0^+} \frac{n-n}{h} == \lim_{h\to 0^+} 0 = 0$$

Since the left- and right-hand limits of f at x = n are not equal, f is not differentiable at x = n

$\therefore$ f is not differentiable in (5, 9).

It is observed that f does not satisfy all the conditions of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for $f(x) = [x]$ for $x \in [5, 9]$

(ii) $f(x) = [x]$ for $x \in [-2, 2]$

It is evident that the given function f (x) is not continuous at every integral point.

In particular, f(x) is not continuous at $x = -2$ and $x = 2$

$\Rightarrow$ f (x) is not continuous in $[-2, 2]$.

Also, $f(-2) = [-2] = -2$ and $f(2) = [2] = 2$

$\therefore f(-2) \neq f(2)$

The differentiability of f in $(-2, 2)$ is checked as follows.

Let n be an integer such that $n \in (-2, 2)$.

The left-hand limit of f at $x = n$ is,

$$\lim_{h\to 0^-} \frac{f(n+h)-f(n)}{h} = \lim_{h\to 0^-} \frac{(n+h)-(n)}{h} = \lim_{h\to 0^-} \frac{n-1-n}{h} == \lim_{h\to 0^-} \frac{-1}{h} = \infty$$

The right-hand limit of f at $x = n$ is,

$$\lim_{h\to 0^+} \frac{f(n+h)-f(n)}{h} = \lim_{h\to 0^+} \frac{(n+h)-(n)}{h} = \lim_{h\to 0^+} \frac{n-n}{h} == \lim_{h\to 0^+} 0 = 0$$

Since the left- and right-hand limits of f at x = n are not equal, f is not differentiable at x = n

$\therefore$ f is not differentiable in $(-2, 2)$.

It is observed that f does not satisfy all the conditions of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for $f(x) = [x]$ for $x \in [-2, 2]$

(iii) $f(x) = x^2 - 1$ for $x \in [1, 2]$

It is evident that f, being a polynomial function, is continuous in [1, 2] and is differentiable in (1, 2).

$f(1) = (1)^2 - 1 = 0$

$f(2) = (2)^2 - 1 = 3$

$\therefore f(1) \neq f(2)$

It is observed that f does not satisfy a condition of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for $f(x) = x^2 - 1$ for $x \in [1, 2]$

Q.3 If $f: [-5, 5] \to R$ is a differentiable function and if $f'(x)$ does not vanish anywhere, then prove that $f(-5) \neq f(5)$

Sol: It is given that $f: [-5, 5] \to R$ is a differentiable function.

Since every differentiable function is a continuous function, we obtain

(a) f is continuous on $[-5, 5]$.

(b) f is differentiable on $(-5, 5)$.

Therefore, by the Mean Value Theorem, there exists $c \in (-5, 5)$ such that

$$f'(c) = \frac{f(5) - f(-5)}{5 - (-5)}$$

$\Rightarrow 10 f'(c) = f(5) - f(-5)$

It is also given that $f'(x)$ does not vanish anywhere.

$\therefore f'(c) \neq 0$

$\Rightarrow 10f'(c) \neq 0$

$\Rightarrow f(5) - f(-5) \neq 0$

$\Rightarrow f(5) \neq f(-5)$

Hence, proved

Q.4 Verify Mean Value Theorem, if $f(x) = x^2 - 4x - 3$ in the interval $[a, b]$, where $a = 1$ and $1 = 4$.

Sol: The given function is $f(x) = x^2 - 4x - 3$

f, being a polynomial function, is continuous in $[1, 4]$ and is differentiable in $(1, 4)$ whose derivative is $2x - 4$.

$f(1) = 1^2 - 4 \times 1 - 3 = -6, f(4) = 4^2 - 4 \times 4 - 3 = -3$

$$\therefore \frac{f(b) - f(a)}{b - a} = \frac{f(4) - f(1)}{4 - 1} = \frac{-3 - (-6)}{3} = \frac{3}{3} = 1$$

Mean Value Theorem states that there is a point $c \in (1, 4)$ such that $f'(c) = 1$

$f'(c) = 1$

$\Rightarrow 2c - 4 = 1$

$\Rightarrow c = \frac{5}{2}$, where $c = \frac{5}{2} \in (1, 4)$

Hence, Mean Value Theorem is verified for the given function

Q.5 Verify Mean Value Theorem, if $f(x) = x^3 - 5x^2 - 3x$ in the interval $[a, b]$, where $a = 1$ and $b = 3$. Find all $c \in (1, 3)$ such that $f'(c) = 0$

Sol: The given function is $f(x) = x^3 - 5x^2 - 3x$

f, being a polynomial function, is continuous in $[1, 4]$ and is differentiable in $(1, 4)$ whose derivative is $3x^2 - 10x - 3$

$f(1) = 1^3 - 5 \times 1^2 - 3 \times 1 = -7,\ \ f(3) = 3^2 - 5 \times 3^2 - 3 \times 3 = -27$

$$\therefore\ \frac{f(b) - f(a)}{b - a} = \frac{f(3) - f(1)}{3 - 1} = \frac{-27 - (-7)}{3 - 1} = -10$$

Mean Value Theorem states that there is a point $c\ \in\ (1, 4)$ such that $f'(\,c\,) = -10$

$f'(\,c\,) = -10$

$\Rightarrow 3c^2 - 10c - 3 = 10$

$\Rightarrow 3c^2 - 10c + 7 = 0$

$\Rightarrow 3c^2 - 3c - 7c + 7 = 0$

$\Rightarrow 3c(c - 1) - 7(c - 1) = 0$

$\Rightarrow (c - 1)(3c - 7) = 0$

$\Rightarrow c = 1, \frac{7}{3}$, where $c = \frac{7}{3} \in (1, 3)$

Hence, Mean Value Theorem is verified for the given function and $c = \frac{7}{3} \in (1, 3)$ is the only point for which $f'(\,c\,) = 0$

Q.6 Examine the applicability of Mean Value Theorem for all three functions given in the above exercise 2.

Sol: Mean Value Theorem states that for a function $f: [a, b] \to R$, if

(a) f is continuous on $[a,\ b]$

(b) f is differentiable on $(a,\ b)$

then, there exists some $c \in\ (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$

Therefore, Mean Value Theorem is not applicable to those functions that do not satisfy any of the two conditions of the hypothesis.

(i) $f(x) = [x]$ for $x \in [5, 9]$

It is evident that the given function f (x) is not continuous at every integral point.

In particular, f(x) is not continuous at x = 5 and x = 9

$\Rightarrow$ f (x) is not continuous in [5, 9]

The differentiability of f in (5, 9) is checked as follows.

Let n be an integer such that n ∈ (5, 9)

The left-hand limit of f at $x = n$ is,

$$\lim_{h \to 0^-} \frac{f(n + h) - f(n)}{h} = \lim_{h \to 0^-} \frac{(n + h) - (n)}{h} = \lim_{h \to 0^-} \frac{n - 1 - n}{h} == \lim_{h \to 0^-} \frac{-1}{h} = \infty$$

The right-hand limit of f at $x = n$ is,

$$\lim_{h \to 0^+} \frac{f(n + h) - f(n)}{h} = \lim_{h \to 0^+} \frac{(n + h) - (n)}{h} = \lim_{h \to 0^+} \frac{n - n}{h} == \lim_{h \to 0^+} 0 = 0$$

Since the left- and right-hand limits of f at $x = n$ are not equal, f is not differentiable at $x = n$

$\therefore$ f is not differentiable in (5, 9).

It is observed that f does not satisfy all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is not applicable for $f(x) = [x]$ for $x \in [5, 9]$

(ii) $f(x) = [x]$ for $x \in [-2, 2]$

It is evident that the given function f (x) is not continuous at every integral point.

In particular, f(x) is not continuous at $x = -2$ and $x = 2$

$\Rightarrow$ f (x) is not continuous in [− 2 , 2].

The differentiability of f in (− 2 , 2) is checked as follows.

Let n be an integer such that $n \in (-2, 2)$.

The left-hand limit of f at $x = n$ is,

$$\lim_{h\to 0^-} \frac{f(n+h) - f(n)}{h} = \lim_{h\to 0^-} \frac{(n+h) - (n)}{h} = \lim_{h\to 0^-} \frac{n-1-n}{h} == \lim_{h\to 0^-} \frac{-1}{h} = \infty$$

The right-hand limit of f at $x = n$ is,

$$\lim_{h\to 0^+} \frac{f(n+h) - f(n)}{h} = \lim_{h\to 0^+} \frac{(n+h) - (n)}{h} = \lim_{h\to 0^+} \frac{n-n}{h} == \lim_{h\to 0^+} 0 = 0$$

Since the left- and right-hand limits of f at $x = n$ are not equal, f is not differentiable at $x = n$

$\therefore$ f is not differentiable in (− 2 , 2).

It is observed that f does not satisfy all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is not applicable for $f(x) = [x]$ for $x \in [-2, 2]$

(iii) $f(x) = x^2 - 1$ for $x \in [1, 2]$

It is evident that f, being a polynomial function, is continuous in [1, 2] and is differentiable in (1, 2).

It is observed that f satisfies all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is applicable for$f(x) = x^2 - 1$ for $x \in [1, 2]$

It can be proved as follows.

$f(1) = (1)^2 - 1 = 0 \quad f(2) = (2)^2 - 1 = 3$

$$\therefore \frac{f(b) - f(a)}{b - a} = \frac{f(2) - f(1)}{2 - 1} = \frac{3 - 0}{1} = 3$$

$f'(x) = 2x$

$\therefore f'(c) = 3$

$\Rightarrow 2c = 3$

$\Rightarrow c = \frac{3}{2} = 1.5$, where $1.5 \in [1, 2]$