SpeedLabs

MATHS

CBSE 12th

TEEVRA EDUTECH PVT. LTD.

Integrals
Exercise 7.10

1. $\int_0^1 \frac{x}{x^2+1}dx$

Ans. $\int_0^1 \frac{x}{x^2+1}dx$

Let $x^2+1=t \Rightarrow 2xdx=dt$

When $x=0, t=1 \Rightarrow$ and when $x=1, t=2$

$$\therefore \int_0^1 \frac{x}{x^2+1}dx = \frac{1}{2}\int_1^2 \frac{dt}{t}$$

$$=\frac{1}{2}\left[\log|t|\right]_1^2$$

$$=\frac{1}{2}[\log 2-\log 1]$$

$$=\frac{1}{2}\log 2$$

2. $\int_0^{\frac{x}{2}} \sqrt{\sin\phi}\cos^5\phi d\phi$

Ans. Let $I=\int_0^{\frac{x}{2}} \sqrt{\sin\phi}\cos^5\phi d\phi =$

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^4\phi\cos\phi d\phi$$

Also let $\sin\phi=t \Rightarrow \cos\phi d\phi=dt$

When $\phi=0, t=0$ and when $\phi=\frac{\pi}{2}, t=1$

$$\therefore I=\int_0^1 \sqrt{t}\left(1-t^2\right)^2 dt$$

$$=\int_0^1 t^{\frac{1}{2}}\left(1+t^4-2t^2\right)dt$$

$$=\int_0^1\left[t^{\frac{1}{2}}+t^{\frac{9}{2}}-2t^{\frac{5}{2}}\right]dt$$

$$= \left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}} + \frac{t^{\frac{11}{2}}}{\frac{11}{2}} - \frac{2t^{\frac{7}{2}}}{\frac{7}{2}} \right]_0^1$$

$$= \frac{2}{3} + \frac{2}{11} - \frac{4}{7}$$

$$= \frac{2}{3} + \frac{2}{11} - \frac{4}{7}$$

$$= \frac{154 + 42 - 132}{231}$$

$$= \frac{64}{231}$$

3. $\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

Ans. Let $I = \int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

Also, let $x = \tan\theta$ A $dx = \sec^2\theta\, d\theta$

When $x = 0, \theta = 0$ and when $x = 1, \theta = \frac{\pi}{4}$

When $\phi = 0, t = 0$ and when $\phi = \frac{\pi}{2}, t = 1$

$$I = \int_0^{\frac{x}{4}} \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)\sec^2\theta d\theta$$

$$= \int_0^{\frac{x}{4}} \sin^{-1}(\sin 2\theta)\sec^2\theta d\theta$$

$$= \int_0^{\frac{x}{4}} 2\theta \cdot \sec^2\theta d\theta$$

$$= 2\int_0^{\frac{x}{4}} \theta \cdot \sec^2\theta d\theta$$

Taking θ as first function and $\sec^2\theta$ as second function and integrating by parts, we obtain

$$I = 2\left[\theta\int \sec^2\theta d\theta - \int\left\{\left(\frac{d}{dx}\theta\right)\int \sec^2\theta d\theta\right\}d\theta\right]_4^{\frac{x}{4}}$$

$$= 2\left[\theta\tan\theta - \int \tan\theta d\theta\right]_4^{\frac{x}{4}}$$

$$= 2\left[\theta\tan\theta + \log|\cos\theta\right]_{4}^{\frac{x}{4}}$$

$$= 2\left[\frac{x}{4}\tan\frac{\pi}{4} + \log\left|\cos\frac{\pi}{4}\right| - \log|\cos 0|\right]$$

$$= 2\left[\frac{\pi}{4} + \log\left(\frac{1}{\sqrt{2}}\right) - \log 1\right]$$

$$= 2\left[\frac{\pi}{4} - \frac{1}{2}\log 2\right]$$

$$= \frac{\pi}{2} - \log 2$$

4. $\int_0^2 x\sqrt{x+2}\left(\text{put } x+2 = t^2\right)$

Ans. $\int_0^2 x\left(x + 2dx\right)$

Let x+2 = t^2 A dx = 2tdt

When x = 0, t = $\sqrt{2}$ and when x = 2, t = 2

$$\therefore \int_0^2 x\sqrt{x+}2dx = \int_{\sqrt{2}}^2 \left(t^2 - 2\right)\sqrt{t^2}\,2tdt$$

$$= 2\int_{\sqrt{2}}^2 \left(t^2 - 2\right)t^2 dt$$

$$= 2\int_{\sqrt{2}}^2 \left(t^4 - 2t^2\right)dt$$

$$= 2\left[\frac{t^5}{5} - \frac{2t^3}{3}\right]_{\sqrt{2}}^2$$

$$= 2\left[\frac{32}{5} - \frac{16}{3} - \frac{4\sqrt{2}}{3}\right]$$

$$= 2\left[\frac{96 - 80 - 12 - \sqrt{2} + 20\sqrt{2}}{15}\right]$$

$$= 2\left[\frac{16 + 8\sqrt{2}}{15}\right]$$

$$= \frac{16\left(2 + \sqrt{2}\right)}{15}$$

$$= \frac{16\sqrt{2}\left(\sqrt{2} + 1\right)}{15}$$

5. $\int_0^{\frac{x}{2}} \frac{\sin x}{1+\cos^2 x}\,dx$

Ans. $\int_0^{\frac{x}{2}} \frac{\sin x}{1+\cos^2 x}\,dx$

Let cos x = t A -sin x dx = dt

When x = 0, t = 1 and when $x = \frac{\pi}{2}, t = 0$

$$\Rightarrow \int_0^{\frac{x}{2}} \frac{\sin x}{1+\cos^2 x}\,dx = -\int_1^0 \frac{dt}{1+t^2}$$

$$= -\left[\tan^{-1} t\right]_1^0$$

$$= -\left[\tan^{-1} 0 - \tan^{-1} 1\right]$$

$$= \left[-\frac{\pi}{4}\right]$$

$$= \frac{\pi}{4}$$

6. $\int_0^2 \frac{dx}{x+4-x^2}$

Ans. $\int_0^2 \frac{dx}{x+4-x^2} = \int_0^2 \frac{dx}{-\left(x^2 - x + \frac{1}{4} - \frac{1}{4} - 4\right)}$

$$= \int_0^2 \frac{dx}{-\left(x^2 - x + \frac{1}{4} - \frac{1}{4} - 4\right)}$$

$$= \int_0^2 \frac{dx}{-\left[\left(x - \frac{1}{2}\right)^2 - \frac{17}{4}\right]}$$

$$= \int_0^2 \frac{dx}{\left(\frac{\sqrt{17}}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2}$$

Let $x - \frac{1}{2} = t$ A dx = dt

When x = 0, t = - $\frac{1}{2}$ and when x = 2, t = $\frac{3}{2}$

$$\therefore \int_0^2 \frac{dx}{\left(\frac{\sqrt{17}}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2} = \int_{-\frac{1}{2}}^{\frac{3}{2}} \frac{dt}{\left(\frac{\sqrt{17}}{2}\right)^2 - t^2}$$

$$= \left[\frac{1}{2\left(\frac{\sqrt{17}}{2}\right)} \log \frac{\frac{\sqrt{17}}{2} + t}{\frac{\sqrt{17}}{2} - t} \right]_{-\frac{1}{2}}^{\frac{3}{2}}$$

$$= \frac{1}{\sqrt{17}} \left[\log \frac{\frac{\sqrt{17}}{2} + \frac{3}{2}}{\frac{\sqrt{17}}{2} - \frac{3}{2}} - \frac{\log \frac{\sqrt{17}}{2} - \frac{1}{2}}{\log \frac{\sqrt{17}}{2} + \frac{1}{2}} \right]$$

$$= \frac{1}{\sqrt{17}} \left[\log \frac{\sqrt{17} + 3}{\sqrt{17} - 3} - \log \frac{\sqrt{17} - 1}{\sqrt{17} + 1} \right]$$

$$= \frac{1}{\sqrt{17}} \log \frac{\sqrt{17} + 3}{\sqrt{17} - 3} \times \frac{\sqrt{17} + 1}{\sqrt{17} - 1}$$

$$= \frac{1}{\sqrt{17}} \log \left[\frac{17 + 3 + 4\sqrt{17}}{17 + 3 - 4\sqrt{17}} \right]$$

$$= \frac{1}{\sqrt{17}} \log \left[\frac{20 + 4\sqrt{17}}{20 - 4\sqrt{17}} \right]$$

$$= \frac{1}{\sqrt{17}} \log \left(\frac{5 + \sqrt{17}}{5 - \sqrt{17}} \right)$$

$$= \frac{1}{\sqrt{17}} \log \left[\frac{\left(5 + \sqrt{17}\right)\left(5 + \sqrt{17}\right)}{25 - 17} \right]$$

$$= \frac{1}{\sqrt{17}} \log \left[\frac{25 + 17 + 10\sqrt{17}}{8} \right]$$

$$= \frac{1}{\sqrt{17}} \log \left(\frac{42 + 10\sqrt{17}}{8} \right)$$

$$= \frac{1}{\sqrt{17}} \log \left(\frac{21 + 5\sqrt{17}}{4} \right)$$

7. $\int_{-1}^{1}\frac{dx}{x^2+2x+5}$

Ans. $\int_{-1}^{1}\frac{dx}{x^2+2x+5}=\int_{-1}^{1}\frac{dx}{\left(x^2+2x+1\right)+4}=\int_{-1}^{1}\frac{dx}{\left(x+1\right)^2+\left(2\right)^2}$

Let x + 1 = t A dx = dt

When x = -1, t = 0 and when x = 1, t = 2

$\therefore \int_{-1}^{1}\frac{dx}{\left(x+1\right)^2+\left(2\right)^2}=\int_{0}^{2}\frac{dt}{t^2+2^2}$

$=\left[\frac{1}{2}\tan^{-1}\frac{t}{2}\right]_0^2$

$=\frac{1}{2}\tan^{-1}1-\frac{1}{2}\tan^{-1}0$

$=\frac{1}{2}\left(\frac{\pi}{4}\right)=\frac{\pi}{8}$

8. $\int_{1}^{2}\left(\frac{1}{x}-\frac{1}{2x^2}\right)e^{2x}dx$

Ans. $\int_{1}^{2}\left(\frac{1}{x}-\frac{1}{2x^2}\right)e^{2x}dx$

Let 2x = t A 2dx = dt

When x = 1, t = 2 and when x = 2, t = 4

$\therefore \int_{1}^{2}\left(\frac{1}{x}-\frac{1}{2x^2}\right)e^{2x}dx=\frac{1}{2}\int_{2}^{4}\left(\frac{2}{t}-\frac{2}{t^2}\right)e^{t}dt$

$=\int_{2}^{4}\left(\frac{1}{t}-\frac{1}{t^2}\right)e^{t}dt$

=

$\text{let}\ \frac{1}{t}=f\left(t\right)$

Then, $\text{let}\ f'\left(t\right)=-\frac{1}{t^2}$

$\Rightarrow \int_{2}^{4}\left(\frac{1}{t}-\frac{1}{t^2}\right)e^{t}dt=\int_{2}^{4}e^{t}\left[f\left(t\right)+f'\left(t\right)\right]dt$

$=\left[e^{t}f\left(t\right)\right]_2^4$

$$=\left[e^t \cdot \frac{2}{t}\right]_2^4$$

$$=\left[\frac{e^t}{t}\right]_2^4$$

$$=\frac{e^4}{4}-\frac{e^2}{2}$$

$$=\frac{e^2\left(e^2-2\right)}{4}$$

9. The value of the integral $\int_{\frac{1}{3}}^{1}\frac{\left(x-x^3\right)^{\frac{1}{3}}}{x^4}dx$ is

(A) 6 (B) 0 (C) 3 (D) 4

Ans. Let $I=\int_{\frac{1}{3}}^{1}\frac{\left(x-x^3\right)^{\frac{1}{3}}}{x^4}dx$

Also, let $x=\sin\theta \Rightarrow dx=\cos\theta d\theta$

When $x=\frac{1}{3}, \theta=\sin^{-1}\left(\frac{1}{3}\right)$ and when $x=1, \theta=\frac{\pi}{2}$

$$\Rightarrow I=\int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}}\frac{\left(\sin\theta-\sin^3\theta\right)^{\frac{1}{3}}}{\sin^4\theta}\cos\theta d\theta$$

$$=\int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}}\frac{(\sin\theta)^{\frac{1}{3}}\left(1-\sin^2\theta\right)^{\frac{1}{3}}}{\sin^4\theta}\cos\theta d\theta$$

$$=\int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}}\frac{(\sin\theta)^{\frac{1}{3}}(\cos\theta)^{\frac{2}{3}}}{\sin^4\theta}\cos\theta d\theta$$

$$=\int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}}\frac{(\sin\theta)^{\frac{1}{3}}(\cos\theta)^{\frac{2}{3}}}{\sin^2\theta\sin^2\theta}\cos\theta d\theta$$

$$=\int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}}\frac{(\cos\theta)^{\frac{5}{3}}}{(\sin\theta)^{\frac{5}{3}}}\operatorname{cosec}\theta d\theta$$

$$=\int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}}(\cot\theta)^{\frac{5}{3}}\operatorname{cosec}^2\theta d\theta$$

Let $\cot\theta = t$ A $-\operatorname{cosec}2\theta\ d\theta = dt$

When $\theta=\sin^{-1}\left(\frac{1}{3}\right), t=2\sqrt{2}$ and when $\theta=\frac{\pi}{2}, t=0$

$$\therefore I=-\int_{2\sqrt{2}}^{0}(t)^{\frac{5}{3}}dt$$

$$=-\left[\frac{3}{8}(t)^{\frac{8}{3}}\right]_{2\sqrt{2}}^{0}$$

$$=-\frac{3}{8}\left[(t)^{\frac{8}{3}}\right]_{2\sqrt{2}}^{0}$$

$$=-\frac{3}{8}\left[-\left(2\sqrt{2}\right)^{\frac{8}{3}}\right]$$

$$=\frac{3}{8}\left[-\left(\sqrt{8}\right)^{\frac{8}{3}}\right]$$

$$=\frac{3}{8}\left[(8)^{\frac{4}{3}}\right]$$

$$=\frac{3}{8}[16]$$

$$=3\times 2$$

$$=6$$

Hence, the correct answer is A.

10. If $f(x)=\int_0^x t\sin t\ dt$, then $f'(x)$ is

(A) COS X + X SIN X (B) X SIN X (C) X COS X (D) SIN X + X COS X

Ans. $f(x)=\int_0^x t\sin t dt$

Integrating by parts, we obtain

$$f(x)=t\int_0^x \sin dt-\int_0^x\left\{\left(\frac{d}{dt}t\right)\int\sin t\ dt\right\}dt$$

$$=\left[t(-\cos t)\right]_0^x-\int_0^x(-\cos t)dt$$

$$=\left[-t\cos t+\sin t\right]_0^x$$

$$=-x\cos x+\sin x$$

$$\Rightarrow f'(x) = -\left[\left\{x(-\sin x)\right\} + \cos x\right] + \cos x$$

$$= x\sin x - \cos x + \cos x + \cos x$$

$$= x\sin x$$

Hence, correct answer is B.