



SpeedLabs
Maths

CBSE 11th

TEEVRA EDUTECH PVT. LTD.

Limits And Derivatives

Chapter 13.2

Question 1: Find the derivative of $x^2 - 2$ at $x = 10$.

Answer:

Let $f(x) = x^2 - 2$. Accordingly,

$$\begin{aligned}f'(10) &= \lim_{h \rightarrow 0} \frac{f(10+h) - f(10)}{h} \\&= \lim_{h \rightarrow 0} \frac{[(10+h)^2 - 2] - (10^2 - 2)}{h} \\&= \lim_{h \rightarrow 0} \frac{10^2 + 2 \cdot 10 \cdot h + h^2 - 2 - 10^2 + 2}{h} \\&= \lim_{h \rightarrow 0} \frac{20h + h^2}{h} \\&= \lim_{h \rightarrow 0} (20 + h) = (20 + 0) = 20\end{aligned}$$

Thus, the derivative of $x^2 - 2$ at $x = 10$ is 20.

Question 2: Find the derivative of $99x$ at $x = 100$.

Answer:

Let $f(x) = 99x$. Accordingly

$$\begin{aligned}f'(100) &= \lim_{h \rightarrow 0} \frac{f(100+h) - f(100)}{h} \\&= \lim_{h \rightarrow 0} \frac{99(100+h) - 99(100)}{h}\end{aligned}$$

$$= \lim_{h \rightarrow 0} \frac{99 \times 100 + 99h - 99 \times 100}{h}$$

$$= \lim_{h \rightarrow 0} \frac{99h}{h}$$

$$= \lim_{h \rightarrow 0} (99) = 99$$

Thus, the derivative of $99x$ at $x = 100$ is 99 .

Question 3: Find the derivative of x at $x = 1$.

Answer:

Let $f(x) = x$. Accordingly,

$$f'(1) = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(1+h) - 1}{h}$$

$$= \lim_{h \rightarrow 0} \frac{h}{h}$$

$$= \lim_{h \rightarrow 0} (1) = 1$$

Thus, the derivative of x at $x = 1$ is 1 .

Question 4: Find the derivative of the following functions from first principle.

(i) $x^3 - 27$ (ii) $(x - 1)(x - 2)$ (iii) $\frac{1}{x^2}$ (iv) $\frac{x+1}{x-1}$

Answer:

(i) Let $f(x) = x^3 - 27$. Accordingly, from the first principle,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{[(x+h)^3 - 27] - (x^3 - 27)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{x^3 + h^3 + 3x^2h + 3xh^2 - x^3}{h}$$

$$= \lim_{h \rightarrow 0} \frac{h^3 + 3x^2h + 3xh^2}{h}$$

$$= \lim_{h \rightarrow 0} (h^2 + 3x^2 + 3xh)$$

$$= 0 + 3x^2 + 0 = 3x^2$$

(ii) Let $f(x) = (x - 1)(x - 2)$. Accordingly, from the first principle,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(x+h-1)(x+h-2) - (x-1)(x-2)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(x^2 + hx - 2x + hx + h^2 - 2h - x - h + 2) - (x^2 - 2x - x + 2)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(hx + hx + h^2 - 2h - h)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{2hx + h^2 - 3h}{h}$$

$$= \lim_{h \rightarrow 0} (2hx + -3)$$

$$= (2x + 0 - 3) = 2x - 3$$

(iii) Let $f(x) = \frac{1}{x^2}$. Accordingly, from the first principle,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\left(\frac{1}{(x+h)^2}\right) - \left(\frac{1}{x^2}\right)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{x^2 - (x+h)^2}{x^2(x+h)^2} \right]$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{x^2 - x^2 - h^2 - 2hx}{x^2(x+h)^2} \right]$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-h^2 - 2hx}{x^2(x+h)^2} \right]$$

$$= \lim_{h \rightarrow 0} \left[\frac{-h - 2hx}{x^2(x+h)^2} \right]$$

$$= \frac{0 - 2x}{x^2(x+0)^2} = -\frac{2}{x^3}$$

(iv) Let $f(x) = \frac{x+1}{x-1}$. Accordingly, from the first principle,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\left(\frac{x+h+1}{x+h-1} - \frac{x+1}{x-1} \right)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{(x^2 + hx + x - x - h - 1) - (x^2 + hx - x + x + h - 1)}{(x-1)(x+h-1)} \right]$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-2h}{(x-1)(x+h-1)} \right]$$

$$= \lim_{h \rightarrow 0} \left[\frac{-2}{(x-1)(x+h-1)} \right]$$

$$= \frac{-2}{(x-1)(x-1)} = \frac{-2}{(x-1)^2}$$

Question 5: For the function

$$f(x) = \frac{x^{100}}{100} + \frac{x^{99}}{99} + \dots + \frac{x^2}{2} + x + 1$$

Prove that $f'(1) = 100f'(0)$

Answer:

The given function is

$$f(x) = \frac{x^{100}}{100} + \frac{x^{100}}{100} + \dots + \frac{x^{100}}{100} + x + 1$$

$$\frac{d}{dx} f(x) = \frac{d}{dx} \left[\frac{x^{100}}{100} + \frac{x^{100}}{100} + \dots + \frac{x^{100}}{100} + x + 1 \right]$$

$$\frac{d}{dx} f(x) = \frac{d}{dx} \left(\frac{x^{100}}{100} \right) + \frac{d}{dx} \left(\frac{x^{99}}{99} \right) + \dots + \frac{d}{dx} \left(\frac{x^2}{2} \right) + \frac{d}{dx} (x) + \frac{d}{dx} (1)$$

On using $\frac{d}{dx} (x^n) = nx^{n-1}$, we obtain

$$\frac{d}{dx} f(x) = \frac{100x^{99}}{100} + \frac{99x^{98}}{99} + \dots + \frac{2x}{2} + 1 + 0$$

$$= x^{99} + x^{98} + \dots + x + 1$$

$$\therefore f'(x) = x^{99} + x^{98} + \dots + x + 1$$

At $x=0$,

$$f'(0) = 1$$

At $x=1$,

$$f'(1) = 1^{99} + 1^{98} + \dots + 1 + 1 = [1 + 1 + \dots + 1 + 1]_{100 \text{ terms.}} = 1 \times 100 = 100$$

$$\text{Thus, } f'(1) = 100 \times f'(0)$$

Question 6: Find the derivative of $x^n + ax^{n-1} + a^2x^{n-2} + \dots + a^{n-1}x + a^n$ for some fixed real number a .

Answer: Let $f(x) = x^n + ax^{n-1} + a^2x^{n-2} + \dots + a^{n-1}x + a^n$

$$\therefore f'(x) = \frac{d}{dx} (x^n + ax^{n-1} + a^2x^{n-2} + \dots + a^{n-1}x + a^n)$$

$$= \frac{d}{dx} (x^n) + a \frac{d}{dx} (x^{n-1}) + a^2 \frac{d}{dx} (x^{n-2}) + \dots + a^{n-1} \frac{d}{dx} (x) + a^n \frac{d}{dx} (1)$$

On Using theorem $\frac{d}{dx}x^n = nx^{n-1}$, we obtain

$$f'(x) = nx^{n-1} + a(n-1)x^{n-2} + a^2(n-2)x^{n-3} + \dots + a^{n-1} + a^n(0)$$
$$= nx^{n-1} + a(n-1)x^{n-2} + a^2(n-2)x^{n-3} + \dots + a^{n-1}$$

Question 7: For some constants a and b, find the derivative of

(i) $(x-a)(x-b)$ (ii) $(ax^2 + b)^2$ (iii) $\frac{x-a}{x-b}$

Answer: Let $f(x) = (x-a)(x-b)$

$$\Rightarrow f(x) = x^2 - (a+b)x + ab$$

$$\therefore f'(x) = \frac{d}{dx}(x^2 - (a+b)x + ab)$$

$$\frac{d}{dx}(x^2) - (a+b)\frac{d}{dx}(x) + \frac{d}{dx}(ab)$$

On using theorem $\frac{d}{dx}(x^n) = nx^{n-1}$, we obtain

$$f'(x) = a^2(4x^3) + 2ab(2x) + b^2(0) = 4a^2x^3 + 4abx = 4ax(ax^2 + b)$$

$$\text{Let } f(x) = \frac{(x-a)}{(x-b)}$$

$$\Rightarrow f'(x) = \frac{d}{dx} \frac{(x-a)}{(x-b)}$$

By quotient rule,

$$f'(x) = \frac{(x-b)\frac{d}{dx}(x-a) - (x-a)\frac{d}{dx}(x-b)}{(x-b)^2} = \frac{(x-b)(1) - (x-a)(1)}{(x-b)^2} = \frac{x-b-x+a}{(x-b)^2} = \frac{a-b}{(x-b)^2}$$

Question 8: Find the derivative of $\frac{x^n - a^n}{x - a}$ for some constant a.

Answer: Let $f(x) = \frac{x^n - a^n}{x - a}$

$$\Rightarrow f'(x) = \frac{d}{dx} \left(\frac{x^n - a^n}{x - a} \right)$$

By quotient rule,

$$f'(x) = \frac{(x - a) \frac{d}{dx} (x^n - a^n) - (x^n - a^n) \frac{d}{dx} (x - a)}{(x - a)^2}$$

$$= \frac{(x - a)(nx^{n-1} - 0) - (x^n - a^n)}{(x - a)^2}$$

$$= \frac{nx^n - anx^{n-1} - x^n + a^n}{(x - a)^2}$$

Question 9: Find the derivative of

(i) $2x - \frac{3}{4}$ (ii) $(5x^3 + 3x - 1)(x - 1)$ (iii) $x^{-3}(3 - 4x^{-5})$

(iv) $x^5(3 - 6x^{-9})$ (v) $x^{-4}(3 - 4x^{-5})$ (vi) $\frac{2}{x+1} - \frac{x^2}{3x-1}$

Answer:

(i) Let $f(x) = 2x - \frac{3}{4}$ $f'(x) = \frac{d}{dx} \left(2x - \frac{3}{4} \right) = 2 \frac{d}{dx} (x) - \frac{d}{dx} \left(\frac{3}{4} \right) = 2 - 0 = 2$

(ii) Let $f(x) = (5x^3 + 3x - 1)(x - 1)$

By Leibnitz product rule,

$$f'(x) = (5x^3 + 3x - 1) \frac{d}{dx} (x - 1) + (x - 1) \frac{d}{dx} (5x^3 + 3 - 0)$$

$$= (5x^3 + 3x - 1)(1) + (x - 1)(5 \cdot 3x^2 + 3 - 0)$$

$$= (5x^3 + 3x - 1)(1) + (x - 1)(15x^2 + 3)$$

$$= 5x^3 + 3x - 1 + 15x^3 + 3x - 15x^2 - 3$$

$$= 20x^3 - 15x^2 + 6x - 4$$

(iii) Let $f(x) = x^{-3}(5 + 3x)$

By Leibnitz product rule,

$$f'(x) = x^{-3} \frac{d}{dx} (5 + 3x) + (5 + 3x) \frac{d}{dx} (x^{-3})$$

$$= x^{-3}(0 + 3) + (5 + 3x)(-3x^{-3-1})$$

$$= x^{-3}(3) + (5 + 3x)(-3x^{-4})$$

$$= 3x^{-3} - 15x^{-4} - 9x^{-3}$$

$$= -6x^{-3} - 15x^{-4}$$

$$= -3x^{-3} \left(2 + \frac{5}{x} \right)$$

$$= -\frac{3x^{-3}}{x} (2x + 5)$$

$$= \frac{-3}{x^4} (5 + 2x)$$

(iv) Let $f(x) = x^5(3 - 6x^{-9})$

By Leibnitz product rule,

$$f'(x) = x^5 \frac{d}{dx} (3 - 6x^{-9}) + (3 - 6x^{-9}) \frac{d}{dx} (x^5)$$

$$= x^5 \{ 0 - 6(-9)x^{-9-1} \} + (3 - 6x^{-9})(5x^4)$$

$$= x^5(54x^{-10}) + 15x^4 - 30x^{-5}$$

$$= 24x^{-5} + 15x^4 = 15x^4 + \frac{24}{x^5}$$

(v) Let $f(x) = x^{-4}(3 - 4x^{-5})$

By Leibnitz product rule,

$$f'(x) = x^{-4} \frac{d}{dx} (3 - 4x^{-5}) + (3 - 4x^{-5}) \frac{d}{dx} (x^{-4})$$

$$= x^{-4} \{0 - 4(-5)x^{-5-1}\} + (3 - 4x^{-5})(-4)x^{-4-1}$$

$$= x^{-4}(20x^{-6}) + (3 - 4x^{-5})(-4x^{-5})$$

$$= 36x^{-10} - 12x^{-5}$$

$$= \frac{12}{x^5} + \frac{36}{x^{10}}$$

(vi) Let $f(x) = \frac{2}{x+1} - \frac{x^2}{3x-1}$

$$f'(x) = \frac{d}{dx} \left(\frac{2}{x+1} \right) - \frac{d}{dx} \left(\frac{x^2}{3x-1} \right)$$

By quotient rule,

$$f'(x) = \left[\frac{(x+1) \frac{d}{dx} (2) - 2 \frac{d}{dx} (x+1)}{(x+1)^2} \right] - \left[\frac{(3x-1) \frac{d}{dx} (x^2) - x^2 \frac{d}{dx} (3x-1)}{(3x-1)^2} \right]$$

$$= \left[\frac{(x+1)(0) - 2(1)}{(x+1)^2} \right] \left[\frac{(3x-1)(2x) - (x^2)(3)}{(3x-1)^2} \right]$$

$$= \frac{-2}{(x+1)^2} - \left[\frac{6x^2 - 2x - 3x^2}{(3x-1)^2} \right]$$

$$= \frac{-2}{(x+1)^2} - \left[\frac{3x^2 - 2x^2}{(3x-1)^2} \right]$$

$$= \frac{-2}{(x+1)^2} - \left[\frac{x(3x-2)}{(3x-1)^2} \right]$$

Question 10: Find the derivative of $\cos x$ from first principle.

Answer: Let $f(x) = \cos x$. accordingly, from the first principle,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos x}{h}$$

$$= \lim_{h \rightarrow 0} \left[\frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \right]$$

$$= \lim_{h \rightarrow 0} \left[\frac{-\cos x(1 - \cos h) - \sin x \sin h}{h} \right]$$

$$= \lim_{h \rightarrow 0} \left[\frac{-\cos x(1 - \cos h)}{h} - \frac{\sin x \sin h}{h} \right]$$

$$= -\cos x \left(\lim_{h \rightarrow 0} \frac{1 - \cos h}{h} \right) - \sin x \lim_{h \rightarrow 0} \left(\frac{\sin h}{h} \right)$$

$$= -\cos x (0) - \sin x (1) = -\sin x \quad \left[\lim_{h \rightarrow 0} \frac{1 - \cos h}{h} = 0 \quad \lim_{h \rightarrow 0} \frac{\sin h}{h} = 1 \right]$$

$$\therefore f'(x) = -\sin x$$

Question 11: Find the derivative of the following functions:

(i) $\sin x \cos x$ (ii) $\sec x$ (iii) $5 \sec x + 4 \cos x$

(iv) $\operatorname{cosec} x$ (v) $3 \cot x + 5 \operatorname{cosec} x$

(vi) $5 \sin x - 6 \cos x + 7$ (vii) $2 \tan x - 7 \sec x$

Answer: (i) Let $f(x) = \sin x \cos x$. accordingly, from the first principle,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sin(x+h) \cos(x+h) - \sin x \cos x}{h}$$

$$= \lim_{h \rightarrow 0} \frac{1}{2h} [2 \sin(x+h) \cos(x+h) - 2 \sin x \cos x]$$

$$= \lim_{h \rightarrow 0} \frac{1}{2h} \left[2 \cos \frac{2x+2h+2x}{2} \cdot \sin \frac{2x+2h-2x}{2} \right]$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[2 \cos \frac{4x+2h}{2} \cdot \sin \frac{2h}{2} \right]$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} [2 \cos(2x+h) \cdot \sin h]$$

$$= \lim_{h \rightarrow 0} \cos(2x+h) \cdot \lim_{h \rightarrow 0} \frac{\sin h}{h} = \cos(2x+0) \cdot 1 = \cos 2x$$

(ii) Let $f(x) = \sec x$. accordingly, from the first principle,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sec(x+h) - \sec(x)}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{1}{\cos(x+h)} - \frac{1}{\cos x} \right]$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\cos x - \cos(x+h)}{\cos x \cos(x+h)} \right]$$

$$= \frac{1}{\cos x} \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-2 \sin \left(\frac{x+x+h}{2} \right) \sin \left(\frac{x-x-h}{2} \right)}{\cos(x+h)} \right]$$

$$= \frac{1}{\cos x} \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-2 \sin \left(\frac{2x+h}{2} \right) \sin \left(-\frac{h}{2} \right)}{\cos(x+h)} \right]$$

$$= \frac{1}{\cos x} \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin \left(\frac{2x+h}{2} \right) \sin \left(\frac{\left(\frac{h}{2} \right)}{\frac{h}{2}} \right)}{\cos(x+h)} \right]$$

$$= \frac{1}{\cos x} \lim_{\frac{h}{2} > 0} \sin\left(\frac{\frac{h}{2}}{\frac{h}{2}}\right) \lim_{h \rightarrow 0} \left(\frac{\sin\left(\frac{2x+h}{2}\right)}{\cos(x+h)}\right)$$

$$= \frac{1}{\cos x} \cdot 1 \cdot \frac{\sin x}{\cos x} = \sec x \tan x$$

(iii) Let $f(x) = 5 \sec x + 4 \cos x$. accordingly, from the first principle,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{5 \sec(x+h) + 4 \cos(x+h) - [5 \sec x + 4 \cos x]}{h}$$

$$= 5 \lim_{h \rightarrow 0} \frac{[\sec(x+h) - \sec x]}{h} + 4 \lim_{h \rightarrow 0} \frac{[\cos(x+h) - \cos x]}{h}$$

$$= 5 \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-2 \sin\left(\frac{x+x+h}{2}\right) \sin\left(\frac{x-x-h}{2}\right)}{\cos(x+h)} \right] + 4 \lim_{h \rightarrow 0} \frac{1}{h} [-\cos x(1 - \cosh) - \sin x \sinh]$$

$$= \frac{5}{\cos x} \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-2 \sin\left(\frac{2x+h}{2}\right) \sin\left(-\frac{h}{2}\right)}{\cos(x+h)} \right] + 4 \left[-\cos x \lim_{h \rightarrow 0} \left(\frac{1 - \cosh}{h}\right) - \sin x \lim_{h \rightarrow 0} \frac{\sinh}{h} \right]$$

$$= \frac{5}{\cos x} \cdot \lim_{h \rightarrow 0} \left[\frac{\sin\left(\frac{2x+h}{2}\right) \cdot \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}}{\cos(x+h)} \right] + 4[(-\cos x) \cdot (0) - (\sin x) \cdot 1]$$

$$= \frac{5}{\cos x} \left[\lim_{h \rightarrow 0} \frac{\sin\left(\frac{2x+h}{2}\right)}{\cos(x+h)} \cdot \lim_{h \rightarrow 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \right] - 4\sin x$$

$$= \frac{5}{\cos x} \cdot \frac{\sin x}{\cos x} \cdot 1 - 4\sin x$$

$$= 5 \sec x \tan x - 4\sin x$$

(iv) Let $f(x) = \operatorname{cosec} x$. accordingly, from the first principle,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{1}{h} [\operatorname{cosec}(x+h) - \operatorname{cosec} x]$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{1}{\sin(x+h)} - \frac{1}{\sin x} \right]$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin x - \sin(x+h)}{\sin(x+h) \sin x} \right]$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{2 \cos\left(\frac{x+x+h}{2}\right) \cdot \sin\left(\frac{x-x-h}{2}\right)}{\sin(x+h) \sin x} \right]$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{2 \cos\left(\frac{2x+h}{2}\right) \sin\left(-\frac{h}{2}\right)}{\sin(x+h) \sin x} \right]$$

$$= \lim_{h \rightarrow 0} \frac{-\cos\left(\frac{2x+h}{2}\right) \cdot \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}}{\sin(x+h) \sin x}$$

$$= \lim_{h \rightarrow 0} \left(\frac{-\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)\sin x} \right) \cdot \lim_{h \rightarrow 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}$$

$$= \left(\frac{-\cos x}{\sin x \sin x} \right) \cdot 1 = -\operatorname{cosec} x \cot x$$

(v) Let $f(x) = 3\cot x + 5\operatorname{cosec} x$. accordingly, from the first principle,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{3\cot(x+h) + 5\operatorname{cosec}(x+h) - 3\cot x - 5\operatorname{cosec} x}{h}$$

$$= 3 \lim_{h \rightarrow 0} \frac{1}{h} [\cot(x+h) - \cot x] + \lim_{h \rightarrow 0} \frac{1}{h} [\operatorname{cosec}(x+h) - \operatorname{cosec} x] \dots (1)$$

$$\text{Now, } \lim_{h \rightarrow 0} \frac{1}{h} [\cot(x+h) - \cot x]$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\cos(x+h)}{\sin(x+h)} - \frac{\cos x}{\sin x} \right]$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\cos(x+h)\sin x - \cos x \sin(x+h)}{\sin x \sin(x+h)} \right]$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x-x-h)}{\sin x \sin(x+h)} \right]$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(-h)}{\sin x \sin(x+h)} \right]$$

$$= \left(\lim_{h \rightarrow 0} \frac{\sin h}{h} \right) \cdot \left(\lim_{h \rightarrow 0} \frac{1}{\sin x \sin(x+h)} \right)$$

$$= -1 \cdot \frac{1}{\sin x \sin(x+0)} = \frac{-1}{\sin^2 x} = -\operatorname{cosec}^2 x \dots (2)$$

$$\begin{aligned}
& \lim_{h \rightarrow 0} \frac{1}{h} [\operatorname{cosec}(x+h) - \operatorname{cosec} x] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{1}{\sin(x+h)} - \frac{1}{\sin x} \right] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin x - \sin(x+h)}{\sin(x+h)\sin x} \right] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{2 \cos\left(\frac{x+x+h}{2}\right) \cdot \sin\left(\frac{x-x-h}{2}\right)}{\sin(x+h)\sin x} \right] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{2 \cos\left(\frac{2x+h}{2}\right) \sin\left(-\frac{h}{2}\right)}{\sin(x+h)\sin x} \right] \\
&\quad - \cos\left(\frac{2x+h}{2}\right) \cdot \frac{\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \\
&= \lim_{h \rightarrow 0} \frac{-\cos\left(\frac{2x+h}{2}\right) \cdot \frac{\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}}{\sin(x+h)\sin x} \\
&= \lim_{h \rightarrow 0} \left(\frac{-\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)\sin x} \right) \cdot \lim_{h \rightarrow 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \\
&= \left(\frac{-\cos x}{\sin x \sin x} \right) \cdot 1 \\
&= -\operatorname{cosec} x \cot x \dots (3)
\end{aligned}$$

From (1), (2), and (3), we obtain

$$f'(x) = -3\operatorname{cosec}^2 x - 5\operatorname{cosec} x \cot x$$

(vi) Let $f(x) = 5\sin x - 6\cos x + 7$. Accordingly, from the first principle,

$$\begin{aligned}
f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \rightarrow 0} \frac{1}{h} [5 \sin(x+h) - 6 \cos(x+h) + 7 - 5 \sin x + 6 \cos x - 7] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} [5 \{ \sin(x+h) - \sin x \} - 6 \{ \cos(x+h) - \cos x \}] \\
&= 5 \lim_{h \rightarrow 0} \frac{1}{h} [\sin(x+h) - \sin x] - 6 \lim_{h \rightarrow 0} \frac{1}{h} [\cos(x+h) - \cos x] \\
&= 5 \lim_{h \rightarrow 0} \frac{1}{h} \left[2 \cos \left(\frac{x+h+x}{2} \right) \sin \left(\frac{x+h-x}{2} \right) \right] - 6 \lim_{h \rightarrow 0} \frac{\cos x \cosh - \sin x \sinh - \cos x}{h} \\
&= 5 \lim_{h \rightarrow 0} \frac{1}{h} \left[2 \cos \left(\frac{2x+h}{2} \right) \sin \frac{h}{2} \right] - 6 \lim_{h \rightarrow 0} \left[\frac{-\cos x(1 - \cosh h) - \sin x \sinh}{h} \right] \\
&= 5 \lim_{h \rightarrow 0} \left[\cos \left(\frac{2x+h}{2} \right) \frac{\sin \frac{h}{2}}{\frac{h}{2}} \right] - 6 \lim_{h \rightarrow 0} \left[\frac{-\cos x(1 - \cosh)}{h} - \frac{\sin x \sinh}{h} \right] \\
&= 5 \left[\lim_{h \rightarrow 0} \cos \left(\frac{2x+h}{2} \right) \right] \left[\lim_{\frac{h}{2} \rightarrow 0} \frac{\sin \frac{h}{2}}{\frac{h}{2}} \right] - 6 \left[(-\cos x) \left(\lim_{h \rightarrow 0} \frac{1 - \cosh}{h} \right) - \sin x \lim_{h \rightarrow 0} \left(\frac{\sinh}{h} \right) \right] \\
&= 5 \cos x \cdot 1 - 6 [(-\cos x) \cdot (0) - \sin x \cdot 1] \\
&= 5 \cos x + 6 \sin x
\end{aligned}$$

(vii) Let $f(x) = 2 \tan x - 7 \sec x$. accordingly, from the first principle,

$$\begin{aligned}
f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \rightarrow 0} \frac{1}{h} [2 \tan(x+h) - 7 \sec(x+h) - 2 \tan x + 7 \sec x] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} [2 \{ \tan(x+h) - \tan x \} - 7 \{ \sec(x+h) - \sec x \}] \\
&= 2 \lim_{h \rightarrow 0} \frac{1}{h} [\tan(x+h) - \tan x] - 7 \lim_{h \rightarrow 0} \frac{1}{h} [\sec(x+h) - \sec x] \\
&= 2 \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x} \right] - 7 \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{1}{\cos(x+h)} - \frac{1}{\cos x} \right]
\end{aligned}$$

$$\begin{aligned}
&= 2 \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x+h) \cos x - \sin x \cos(x+h)}{\cos x \cos(x+h)} \right] - 7 \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\cos x - \cos(x+h)}{\cos x \cos(x+h)} \right] \\
&= 2 \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x+h-x)}{\cos x \cos(x+h)} \right] - 7 \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-2 \sin\left(\frac{x+x+h}{2}\right) \sin\left(\frac{x-x-h}{2}\right)}{\cos x \cos(x+h)} \right] \\
&= 2 \lim_{h \rightarrow 0} \left[\left(\frac{\sinh}{h} \right) \frac{1}{\cos x \cos(x+h)} \right] - 7 \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-2 \sin\left(\frac{2x+h}{2}\right) \sin\left(-\frac{h}{2}\right)}{\cos x \cos(x+h)} \right] \\
&= 2 \left(\lim_{h \rightarrow 0} \frac{\sinh}{h} \right) \left(\lim_{h \rightarrow 0} \frac{1}{\cos x \cos(x+h)} \right) - 7 \left(\lim_{\frac{h}{2} \rightarrow 0} \frac{\sin \frac{h}{2}}{\frac{h}{2}} \right) \left(\lim_{h \rightarrow 0} \frac{\sin\left(\frac{2x+h}{2}\right)}{\cos x \cos(x+h)} \right) \\
&= 2.1. \frac{1}{\cos x \cos x} - 7.1 \left(\frac{\sin x}{\cos x \cos x} \right) \\
&= 2 \sec^2 x - 7 \sec x \tan x
\end{aligned}$$