SpeedLabs

MATHS

CBSE 12th

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1. Show that the function f: $R_* \to R_*$ defined by $f(x) = \frac{1}{x}$ is one-one and onto, where R* is the set of all non-zero real numbers. Is the result true, if the domain R_* is replaced by N with co-domain being same as R_*?

Ans. It is given that f: $R_* \to R_*$ is defined by $f(x) = \frac{1}{x}$.

One-one:

$f(x) = f(y)$

$\Rightarrow \frac{1}{x} = \frac{1}{y}$

$\Rightarrow x = y$

$\therefore$f is one-one.

Onto:

It is clear that for y$\in R_*$, there exists $x = \frac{1}{y} \notin R.(\text{Existsasy} \neq 0)$ such that

$f(x) = \frac{1}{\left(\frac{1}{y}\right)} = y.$

$\therefore$f is onto.

Thus, the given function (f) is one-one and onto.

Now, consider function g: N $\to R_*$ defined by

$g(x) = \frac{1}{x}$.

We have,

$g(x_1) = g(x_2) \Rightarrow \frac{1}{x_1} = \frac{1}{x_2} \Rightarrow x_1 = x_2$

$\therefore$g is one-one.

Further, it is clear that g is not onto as for 1.2 $\in R_*$ there does not exit any x in **N** such that $g(x) = \frac{1}{1.2}$

Hence, function g is one-one but not onto.

2. Check the injectivity and surjectivity of the following functions:

(i) f: **N → N** given by $f(x) = x^2$

(ii) f: **Z → Z** given by $f(x) = x^2$

(iii) f: **R** → **R** given by $f(x) = x^2$

(iv) f: **N** → **N** given by $f(x) = x^3$

(v) f: **Z** → **Z** given by $f(x) = x^3$

Ans. (i) f: N → N is given by,

$f(x) = x^2$

It is seen that for x, y ∈N, $f(x) = f(y) \Rightarrow x^2 = y^2 \Rightarrow x = y$.

∴ f is injective.

Now, 2 ∈ **N**. But, there does not exist any x in N such that $f(x) = x^2 = 2$.

∴ f is not surjective.

Hence, function f is injective but not surjective.

(ii) f: **Z** → **Z** is given by,

$f(x) = x^2$

It is seen that $f(-1) = f(1) = 1$, but $-1 \neq 1$.

∴ f is not injective.

Now,−2 ∈ **Z.** But, there does not exist any element x ∈Z such that $f(x) = x^2 = -2$.

∴ f is not surjective.

Hence, function f is neither injective nor surjective.

(iii) f: **R** → **R** is given by,

$f(x) = x^2$

It is seen that $f(-1) = f(1) = 1$, but $-1 \neq 1$.

∴ f is not injective.

Now,−2 ∈ **R**. But, there does not exist any element x ∈ **R** such that $f(x) = x^2 = -2$.

∴ f is not surjective.

Hence, function f is neither injective nor surjective.

(iv) f**: N** → **N** given by,

$f(x) = x^3$

It is seen that for x, y ∈**N**, $f(x) = f(y) \Rightarrow x3 = y3 \Rightarrow x = y$.

∴f is injective.

Now, 2 ∈ **N**. But, there does not exist any element x in domain N such that $f(x) = x^3 = 2$.

∴ f is not surjective

Hence, function f is injective but not surjective.

(v) f**: Z** → **Z** is given by,

$f(x) = x^3$

It is seen that for x, y ∈ Z, $f(x) = f(y) \Rightarrow x^3 = y^3 \Rightarrow x = y$.

∴ f is injective.

Now, $2 \in \mathbf{Z}$. But, there does not exist any element x in domain **Z** such that $f(x) = x^3 = 2$.

$\therefore$ f is not surjective.

Hence, function f is injective but not surjective.

3. Prove that the Greatest Integer Function f: $\mathbf{R} \to \mathbf{R}$ given by f(x) = [x], is neither oneonce nor onto, where [x] denotes the greatest integer less than or equal to x.

Ans. f: $\mathbf{R} \to \mathbf{R}$ is given by,

f(x) = [x]

It is seen that f(1.2) = [1.2] = 1, f(1.9) = [1.9] = 1.

$\therefore$ f(1.2) = f(1.9), but $1.2 \neq 1.9$.

$\therefore$ f is not one-one.

Now, consider $0.7 \in \mathbf{R}$.

It is known that f(x) = [x] is always an integer. Thus, there does not exist any element x $\in \mathbf{R}$ such that f(x) = 0.7.

$\therefore$ f is not onto.

Hence, the greatest integer function is neither one-one nor onto.

4. Show that the Modulus Function f: $\mathbf{R} \to \mathbf{R}$ given by $f(x) = |x|$, is neither one-one nor onto, where $|x|$ is x, if x is positive or 0 and $|x|$ is − x, if x is negative.

And. f: R → R is given by,

$$f(x) = |x| = \begin{cases} x, \text{if} x \geq 0 \\ -x, \text{if} x < 0 \end{cases}$$

It is seen that $f(-1) = |-1| = 1, f(1) = |1| = 1$

$\therefore f(-1) = f(1)$, but $-1 \neq 1$.

$\therefore$ f is not one-one.

Now, consider $-1 \in \mathbf{R}$.

It is known that $f(x) = |x|$ is always non-negative. Thus, there does not exist any element x in domain **R** such that $f(x) = |x| = -1$.

$\therefore$ f is not onto.

Hence, the modulus function is neither one-one nor onto.

5. Show that the Signum Function f: $\mathbf{R} \to \mathbf{R}$, given by

$$f(x) = \begin{cases} 1, \text{if } x > 0 \\ 0, \text{if } x = 0 \\ -1, \text{if} x < 0 \end{cases}$$

is neither one-one nor onto.

Ans. f: $\mathbf{R} \to \mathbf{R}$ is given by,

$$f(x) = \begin{cases} 1, \text{if } x > 0 \\ 0, \text{if } x = 0 \\ -1, \text{if} x < 0 \end{cases}$$

It is seen that $f(1) = f(2) = 1$, but $1 \neq 2$.

∴f is not one-one.

Now, as f(x) takes only 3 values (1, 0, or −1) for the element −2 in co-domain R, there does not exist any x in domain R such that $f(x) = -2$.

∴ f is not onto.

Hence, the signum function is neither one-one nor onto.

6. Let A = {1, 2, 3}, B = {4, 5, 6, 7} and let f = {(1, 4), (2, 5), (3, 6)} be a function from A to B. Show that f is one-one.

Ans. It is given that A = {1, 2, 3}, B = {4, 5, 6, 7}.

f: A → B is defined as f = {(1, 4), (2, 5), (3, 6)}.

∴ f (1) = 4, f (2) = 5, f (3) = 6

It is seen that the images of distinct elements of A under f are distinct.

Hence, function f is one-one.

7. In each of the following cases, state whether the function is one-one, onto or bijective. Justify your answer.

(i) f: **R → R** defined by $f(x) = 3 - 4x$

(ii) f: **R → R** defined by $f(x) = 1 + x^2$

Ans. (i) f: R → R is defined as $f(x) = 3 - 4x$.

Let $x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$

$\Rightarrow 3 - 4x_1 = 3 - 4x_2$

$\Rightarrow -4x = -4x_2$

$\Rightarrow x_1 = x_2$

∴ f is one-one.

For any real number (y) in **R,** there exists $\frac{3-y}{4}$ in **R** such that

$$f\left(\frac{3-y}{4}\right) = 3 - 4\left(\frac{3-y}{4}\right) = y.$$

∴f is onto.

Hence, f is bijective.

(ii) f: **R → R** is defined as

$f(x) = 1 + x^2$.

Let $x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$

$\Rightarrow 1 + x_1^2 = 1 + x_1^2$

$\Rightarrow\ x_1^2 = x_2^2$

$\Rightarrow x_1 = \pm x_2$

$\therefore f(x_1) = f(x_2)$ does not imply that $x_1 = x_2$.

For instance,

$f(1) = f(-1) = 2$

$\therefore$ f is not one-one.

Consider an element -2 in co-domain **R**.

It is seen that $f(x) = 1 + x^2$ is positive for all $x \in$ **R**.

Thus, there does not exist any x in domain **R** such that $f(x) = -2$.

$\therefore$ f is not onto.

Hence, f is neither one-one nor onto.

8. Let A and B be sets. Show that f: $A \times B \rightarrow B \times A$ such that $(a, b) = (b, a)$ is bijective function.

Ans. f: $A \times B \rightarrow B \times A$ is defined as $f(a, b) = (b, a)$.

Let $(a_1, b_1), (a_2, b_2) \in A \times B$ such that $f(a_1, b_1) = f(a_2, b_2)$.

$\Rightarrow (b_1, a_1) = (b_2, a_2)$

$\Rightarrow b_1 = b_2$ and $a_1 = a_2$

$\Rightarrow (a_1, b_2) = (a_2, b_2)$

$\therefore$ f is one-one.

Now, let $(b, a) \in B \times A$ be any element.

Then, there exists $(a, b) \in A \times B$ such that $f(a, b) = (b, a)$. [By definition of f]

$\therefore$ f is onto.

Hence, f is bijective.

9. Let f: **N → N** be defined by

$$f(n) = \begin{cases} \dfrac{n+1}{2}, & \text{if n is odd} \\ \dfrac{n}{2}, & \text{if n is even} \end{cases} \quad \text{for all } n \in \mathbf{N}.$$

State whether the function f is bijective. Justify your answer.

Ans. f: **N → N** is defined as

$$(n) = \begin{cases} \dfrac{n+1}{2}, & \text{if n is odd} \\ \dfrac{n}{2}, & \text{if n is even} \end{cases} \quad \text{for all } n \in \mathbf{N}.$$

It can be observed that :

$$f(1) = \frac{1+1}{2} + \text{and } f(2) = \frac{2}{2} = 1 \text{ [By definition of f]}$$

$\therefore f(1) = f(2)$, where $1 \neq 2$.

$\therefore$ f is not one-one.

Consider a natural number (n) in co-domain **N**.

Case **I**: n is odd

$\therefore$n = 2r + 1 for some r ∈ **N.** Then, there exists 4r + 1∈**N** such that

$f(4r+1) = \frac{4r+1+1}{2}\ 2r+1$

Case **II:** n is even

$\therefore$n = 2r for some r ∈ **N.** Then, there exists 4r ∈**N** such that. $f(4r) = \frac{4r}{2} = 2r$

$\therefore$ f is onto.

Hence, f is not a bijective function.

10. Let A = **R** − {3} and B = **R** − {1}. Consider the function f: A → B defined by

$F(x) = \frac{x-2}{x-3}$. Is f one-one and onto? Justify your answer.

Ans. A = R − {3}, B = R − {1}

Let x, y ∈ A such that f f(x) = f (y).

$\Rightarrow \frac{x-2}{x-3} = \frac{y-2}{y-3}$

$\Rightarrow (x-2)(y-3) = (y-2)(x-3)$

$\Rightarrow xy - 3x - 2y + 6 = xy - 3y - 2x + 6$

$\Rightarrow -3x - 2y = -3y - 2x$

$\Rightarrow 3x - 2x = 3y - 2y$

$\Rightarrow x = y$

$\therefore$ f is one-one.

Let y ∈B = R − {1}. Then, y ≠ 1.

The function f is onto if there exists x ∈A such that f(x) = y.

Now,

$f(x) = y$

$\Rightarrow \frac{x-2}{x-3} = y$

$\Rightarrow x - 2 = xy - 3y$

$\Rightarrow (1 - y) = -3y + 2$

$\Rightarrow x = \frac{2-3y}{1-y} \in A$ [y ≠ 1]

Thus, for any y ∈ B, there exists $\frac{2-3y}{1-y} \in A$ such that

$$f\frac{2-3y}{1-y}=\frac{\left(\frac{2-3y}{1-y}\right)-2}{\left(\frac{2-3y}{1-y}\right)-3}=\frac{2-3y-2+2y}{2-3y-3+3y}=\frac{-y}{-1}=y$$

∴ f is ont.

11. Let f: **R** → **R** be defined as $f(x) = x^4$. Choose the correct answer.

(A) f is one-one onto (B) f is many-one onto

(C) f is one-one but not onto (D) f is neither one-one nor onto

Ans. f: **R** → **R** is defined as $f(x)=x^4$.

Let x, y ∈ **R** such that f(x) = f(y).

$\Rightarrow x^4 = y^4$

$\Rightarrow x = \pm y$

∴ $f(x_1) = f(x_2)$ dose not imply that $x_1 = x_2$.

For instance,

f (1) = f (-1) = 1

∴ f is not one-one.

Consider an element 2 in co-domain **R**. It is clear that there does not exist any x in domain **R** such that f(x) = 2.

∴ f is not onto.

Hence, function f is neither one-one nor onto.

The correct answer is D.

12. Let f: R → R be defined as f(x) = 3x. Choose the correct answer.

(A) f is one-one onto (B) f is many-one onto

(C) f is one-one but not onto (D) f is neither one-one nor onto

Ans. f: R → R is defined as f(x) = 3x.

Let x, y ∈ R such that f(x) = f(y).

$\Rightarrow 3x = 3y$

$\Rightarrow x = y$

∴ f is one-one.

Also, for any real number (y) in co-domain **R**, there exists $\frac{y}{3}$ in **R** such that $f\left(\frac{y}{3}\right)=3\left(\frac{y}{3}\right)=y$

∴ f is onto.

Hence, function f is one-one and onto.

The correct answer is A.