SpeedLabs

MATHS

CBSE 12th

TEEVRA EDUTECH PVT. LTD.

1. Show that the three lines with direction cosines

$\frac{12}{13},\frac{-3}{13},\frac{-4}{13};\frac{4}{13},\frac{12}{13},\frac{3}{13};\frac{3}{13},\frac{-4}{13},\frac{12}{13}$

Ans. Two lines with direction cosines, l_1, m_1, n_1 and l_2, m_2, n_2, are perpendicular to each other, if $l_1l_2 + m_1m_2 + n_1n_2 = 0$

(i) For the lines with direction cosines, $\frac{12}{13},\frac{-3}{13},\frac{-4}{13}$ and $\frac{4}{13},\frac{12}{13},\frac{3}{13}$, we obtain

$$l_1l_2 + m_1m_2 + n_1n_2 = \frac{12}{13}\times\frac{4}{13}+\left(\frac{-3}{13}\right)\times\frac{12}{13}+\left(\frac{-4}{13}\right)\times\frac{3}{13}$$

$$=\frac{48}{169}-\frac{36}{169}-\frac{12}{169}$$
$$=0$$

Therefore, the lines are perpendicular.

(ii) For the lines with direction cosines, $\frac{4}{13},\frac{12}{13},\frac{3}{13}$ and $\frac{3}{13},\frac{-4}{13},\frac{12}{13}$ we obtain,

$$l_1l_2 + m_1m_2 + n_1n_2 = \frac{4}{13}\times\frac{3}{13}+\frac{12}{13}\times\left(\frac{-4}{13}\right)+\frac{3}{13}\times\frac{12}{13}$$

$$=\frac{12}{169}-\frac{48}{169}+\frac{36}{169}$$
$$=0$$

Therefore, the lines are perpendicular.

(iii) For the lines with direction cosines, $\frac{3}{13},\frac{-4}{13},\frac{12}{13}$ *and* $\frac{12}{13},\frac{-3}{13},\frac{-4}{13}$, we obtain,

$$l_1l_2 + m_1m_2 + n_1n_2 = \left(\frac{3}{13}\right)\times\left(\frac{12}{13}\right)+\left(\frac{-4}{13}\right)\times\left(\frac{-3}{13}\right)+\left(\frac{12}{13}\right)\times\left(\frac{-4}{13}\right)$$

$$\frac{36}{169}+\frac{12}{169}-\frac{48}{169}$$
$$=0$$

Therefore, the lines are perpendicular.

Thus, all the lines are mutually perpendicular.

2. Show that the line through the points (1, −1, 2) (3, 4, −2) is perpendicular to the line through the points (0, 3, 2) and (3, 5, 6).

Ans. Let AB be the line joining the points, (1, −1, 2) and (3, 4, − 2), and CD be the line joining the points, (0, 3, 2) and (3, 5, 6). The direction ratios, a_1, b_1, c_1, of AB are (3 − 1), (4 − (−1)), and (−2 − 2) i.e., 2, 5, and −4. The direction ratios, a_2, b_2, c_2, of CD are (3 − 0), (5 − 3), and (6 −2) i.e., 3, 2, and 4. AB and CD will be perpendicular to each other, if $a_1a_2 + b_1b_2 + c_1c_2 = 0$ $a_1a_2 + b_1b_2 + c_1c_2 = 2 \times 3 + 5 \times 2 + (-4) \times 4 = 6 + 10 - 16 = 0$ Therefore, AB and CD are perpendicular to each other.

3. Show that the line through the points (4, 7, 8) (2, 3, 4) is parallel to the line through the points (−1, −2, 1), (1, 2, 5).

Ans. Let AB be the line through the points, (4, 7, 8) and (2, 3, 4), and CD be the line through the points, (−1, −2, 1) and (1, 2, 5). The directions ratios, a_1, b_1, c_1, of AB are (2 − 4), (3 − 7), and (4 − 8) i.e., −2, −4, and −4. The direction ratios, a_2, b_2, c_2, of CD are (1 − (−1)), (2 − (−2)), and (5 − 1) i.e., 2, 4, and 4.

AB will be parallel to CD, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

$$\frac{a_1}{a_2} = \frac{-2}{2} = -1$$

$$\frac{b_1}{b_2} = \frac{-4}{4} = -1$$

$$\frac{c_1}{c_2} = \frac{-4}{4} = -1$$

$$\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Thus, AB is parallel to CD.

4. Find the equation of the line which passes through the point (1, 2, 3) and is parallel to the vector $3\hat{i} + 2\hat{j} - 2\hat{k}$

Ans. It is given that the line passes through the point A (1, 2, 3). Therefore, the position vector through A is

$$\vec{a} + \hat{i} + 2\hat{j} - 2\hat{k}$$

$$\vec{b} + 3\hat{i} + 2\hat{j} - 2\hat{k}$$

It is known that the line which passes through point A and parallel to $\vec{b}$ is given by

$\vec{r} = \vec{a} + \lambda\vec{b}$, where λ is a constant.

$$\Rightarrow \vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \lambda\left(3\hat{i} + 2\hat{j} - 2\hat{k}\right)$$

This is the required equation of the line.

5. Find the equation of the line in vector and in Cartesian form that passes through the point with position vector $2\hat{i} - \hat{j} + 4\hat{k}$ and is in the direction $\hat{i} - 2\hat{j} - \hat{k}$

Ans. It is given that the line passes through the point with position vector

$\vec{a} = \hat{i} - \hat{j} + 4\hat{k}........(1)$

$\vec{b} = \hat{i} + 2\hat{j} - \hat{k}........(2)$

It is known that a line through a point with position vector $\vec{a}$ and parallel to $\vec{b}$ is given by the equation,

$\vec{r} = \vec{a} + \lambda\vec{b}$

$\Rightarrow \vec{r} = 2\hat{i} - \hat{j} + 4\hat{k} + \lambda\left(\hat{i} + 2\hat{j} - \hat{k}\right)$

This is the required equation of the line in vector form.

$\vec{r} = x\hat{i} - y\hat{j} + z\hat{k}$

$\Rightarrow x\hat{i} - y\hat{j} + z\hat{k} = (\lambda + 2)\hat{i} + (2\lambda - 1)\hat{j} + (-\lambda + 4)\hat{k}$

Eliminating λ, we obtain the Cartesian form equation as

$$\frac{x-2}{1} = \frac{y+1}{2} = \frac{z-4}{-1}$$

This is the required equation of the given line in Cartesian form.

6. Find the Cartesian equation of the line which passes through the point (−2, 4, −5) and parallel to the line given by $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$

Ans. It is given that the line passes through the point (−2, 4, −5) and is parallel to

$$\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$$

The direction ratios of the line, $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$ are 3, 5, and 6.

The required line is parallel to $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$

Therefore, its direction ratios are 3k, 5k, and 6k, where k ≠ 0 It is known that the equation of the line through the point (x_1, y_1, z_1) and with direction

ratios, a, b, c, is given by $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$

Therefore, the equation of the required line is

$$\frac{x+2}{3k} = \frac{y-4}{5k} = \frac{z+5}{6k}$$

$$\Rightarrow \frac{x+2}{3} = \frac{y-4}{5} = \frac{z+5}{6} = k$$

7. The Cartesian equation of a line is $\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}$ Write its vector form.

Ans. The Cartesian equation of the line is

$$\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2} \qquad .... (1)$$

The given line passes through the point (5, −4, 6). The position vector of this point is

$\vec{a}=5\hat{i}-4\hat{j}+6\hat{k}$

Also, the direction ratios of the given line are 3, 7, and 2.

This means that the line is in the direction of vector, $\vec{b}=3\hat{i}-7\hat{j}+2\hat{k}$

It is known that the line through position vector $\vec{a}$ and in the direction of the vector $\vec{b}$ is

given by the equation, $\vec{r}=\vec{a}+\lambda\vec{b}, \lambda\in R$

$$\Rightarrow \vec{r}=\left(5\hat{i}-4\hat{j}+6\hat{k}\right)+\lambda\left(3\hat{i}+7\hat{j}+2\hat{k}\right)$$

This is the required equation of the given line in vector form.

8. Find the vector and the Cartesian equations of the lines that pass through the origin and (5, −2, 3).

Ans. The required line passes through the origin. Therefore, its position vector is given by,

$\vec{a}=\vec{0}......(1)$

The direction ratios of the line through origin and (5, −2, 3) are (5 − 0) = 5, (−2 − 0) = −2, (3 − 0) = 3

The line is parallel to the vector given by the equation, $\vec{b}=5\hat{i}-2\hat{j}+3\hat{k}$

The equation of the line in vector form through a point with position vector $\vec{a}$ and parallel

to $\vec{b}$ is, $\vec{r}=\vec{a}+\lambda\vec{b}, \lambda\in R$

$$\Rightarrow \vec{r}=\vec{0}+\lambda\left(5\hat{i}-2\hat{j}+3\hat{k}\right)$$

$$\Rightarrow \vec{r}=\lambda\left(5\hat{i}-2\hat{j}+3\hat{k}\right)$$

The equation of the line through the point (x_1, y_1, z_1) and direction ratios a, b, c is given

by, $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$

Therefore, the equation of the required line in the Cartesian form is

$$\frac{x-0}{5}=\frac{y-0}{-2}=\frac{z-0}{3}$$

$$\Rightarrow \frac{x}{5}=\frac{y}{-2}=\frac{z}{3}$$

9. Find the vector and the Cartesian equations of the line that passes through the points (3, −2, −5), (3, −2, 6).

Ans. Let the line passing through the points, P (3, −2, −5) and Q (3, −2, 6), be PQ. Since PQ passes through P (3, −2, −5), its position vector is given by, $\vec{a}=3\hat{i}-2\hat{j}-5\hat{k}$

The direction ratios of PQ are given by, $(3-3)=0$, $(-2+2)=0$, $(6+5)=11$ The equation of the vector in the direction of PQ is

$$\vec{b}=0.\hat{i}-0.\hat{j}+11\hat{k}=11\hat{k}$$

The equation of PQ in vector form is given by $\vec{r}=\vec{a}+\lambda\vec{b},\lambda\in R$

$$\Rightarrow\vec{r}=\left(3\hat{i}-2\hat{j}-5\hat{k}\right)+11\lambda\hat{k}$$

The equation of PQ in Cartesian form is

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}\text{ i.e,}$$

$$\frac{x-3}{0}=\frac{y+2}{0}=\frac{z+5}{11}$$

10. Find the angle between the following pairs of lines:

$$\vec{r}=2\hat{i}-5\hat{j}+\hat{k}+\lambda\left(3\hat{i}-2\hat{j}+6\hat{k}\right)\text{and}$$

$$(i)\,\vec{r}=7\hat{i}-6\hat{k}+\mu\left(\hat{i}+2\hat{j}+2\hat{k}\right)$$

$$(ii)\,\vec{r}=3\hat{i}+\hat{j}+\mu\left(\hat{i}-\hat{j}-2\hat{k}\right)\text{and}$$

$$\vec{r}=2\hat{i}-\hat{j}-56\hat{k}+\mu\left(3\hat{i}-5\hat{j}-4\hat{k}\right)$$

Ans. (i) Let Q be the angle between the given lines.

The angle between the given pairs of lines is given by, $\cos Q=\left|\frac{\vec{b}_1.\vec{b}_2}{\left|\vec{b}_1\right|\left|\vec{b}_2\right|}\right|$

The given lines are parallel to the vectors, $\vec{b}_1=3\hat{i}+2\hat{j}+6\hat{k}$ and $\vec{b}_2=\hat{i}+2\hat{j}+2\hat{k}$ respectively.

$$\therefore\left|\vec{b}_1\right|=\sqrt{3^2+2^2+6^2}=7$$

$$\left|\vec{b}_2\right|=\sqrt{(1)^2+(2)^2+(2)^2}=3$$

$$\vec{b}_1.\vec{b}_2=\left(3\hat{i}+2\hat{j}+6\hat{k}\right).\left(\hat{i}+2\hat{j}+2\hat{k}\right)$$

$$=3\times1+2\times2+6\times2$$

$$=3+4+12$$

$$=19$$

$$\Rightarrow\cos Q=\frac{19}{7\times3}$$

$$\Rightarrow Q=\cos^{-1}\left(\frac{19}{21}\right)$$

(ii) The given lines are parallel to the vectors, $\vec{b}_1 = \hat{i} - \hat{j} - 2\hat{k}$ and $\vec{b}_2 = 3\hat{i} - 5\hat{j} - 4\hat{k}$ respectively.

$$\therefore \left|\vec{b}_1\right| = \sqrt{(1)^2 + (-1)^2 + (-2)^2} = \sqrt{6}$$

$$\left|\vec{b}_2\right| = \sqrt{(3)^2 + (-5)^2 + (-4)^2} = \sqrt{50} = 5\sqrt{2}$$

$$\vec{b}_1.\vec{b}_2 = \left(\hat{i} - \hat{j} - 2\hat{k}\right).\left(3\hat{i} - 5\hat{j} - 4\hat{k}\right)$$

$$= 1.3 - 1(-5) - 2(-4)$$

$$= 3 + 5 + 8$$

$$= 16$$

$$\cos Q = \frac{16}{\sqrt{6}.5\sqrt{2}} = \frac{16}{\sqrt{2}.\sqrt{3}.5\sqrt{2}} = \frac{16}{10\sqrt{3}}$$

$$\Rightarrow \cos Q = \frac{8}{5\sqrt{3}}$$

$$\Rightarrow Q = \cos^{-1}\left(\frac{8}{5\sqrt{3}}\right)$$

11. Find the angle between the following pairs of lines:

(i) $\frac{x-2}{2} = \frac{y-1}{5} = \frac{z+3}{-3}$ and $\frac{x+2}{-1} = \frac{y-4}{8} = \frac{z-5}{4}$

(ii) $\frac{x}{2} = \frac{y}{2} = \frac{z}{1}$ and $\frac{x-5}{4} = \frac{y-2}{1} = \frac{z-3}{8}$

Ans. Let $\vec{b}_1$ and $\vec{b}_2$ be the vectors parallel to the pair of lines,

$\frac{x-2}{2} = \frac{y-1}{5} = \frac{z-3}{-3}$ and $\frac{x+2}{-1} = \frac{y-4}{8} = \frac{z-5}{4}$, respectively.

$$\therefore \vec{b}_1 = 2\hat{i} + 5\hat{j} - 3\hat{k} \text{ and } \vec{b}_2 = -\hat{i} + 8\hat{j} + 4\hat{k}$$

$$\left|\vec{b}_1\right| = \sqrt{(2)^2 + (5)^2 + (-3)^2} = \sqrt{38}$$

$$\left|\vec{b}_2\right| = \sqrt{(-1)^2 + (8)^2 + (4)^2} = \sqrt{81} = 9$$

$$\vec{b}_1.\vec{b}_2 = \left(2\hat{i} + 2\hat{j} + 8^2\right).\left(4\hat{i} + \hat{j} + 8\hat{k}\right)$$

$$= 2\times4 + 2\times1 + 1\times8$$

$$= 8 + 2 + 8$$

$$= 18$$

If Q is the angle between the given pair of lines, then $\cos Q = \left|\frac{\vec{b}_1.\vec{b}_2}{\left|\vec{b}_1\right|\left|\vec{b}_2\right|}\right|$

$$\Rightarrow \cos Q = \frac{18}{3\times 9} = \frac{2}{3}$$

$$\Rightarrow Q = \cos^{-1}\left(\frac{2}{3}\right)$$

12. Find the values of p so the line $\frac{1-x}{3} = \frac{7y-14}{2p} = \frac{z-3}{2}$ and $\frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5}$ are at right angles.

Ans. The given equations can be written in the standard form as $\frac{x-1}{-3} = \frac{y-2}{\frac{2p}{7}} = \frac{z-3}{2}$ and $\frac{x-1}{\frac{-3p}{7}} = \frac{y-5}{1} = \frac{z-6}{-5}$

The direction ratios of the lines are $-3, \frac{2p}{7}, 2$ and $\frac{-3p}{7}, 1 - 5$ respectively.

Two lines with direction ratios, a_1, b_1, c_1 and a_2, b_2, c_2, are perpendicular to each other, if $a_1a_2 + b_1 b_2 + c_1c_2 = 0$

$$\therefore (-3).\left(\frac{-3p}{7}\right) + \left(\frac{2p}{7}\right).(1) + 2.(-5) = 0$$

$$\Rightarrow \frac{9p}{7} + \frac{2p}{7} = 10$$

$$\Rightarrow 11p = 70$$

$$\Rightarrow p = \frac{70}{11}$$

Thus, the value of p is $\frac{70}{11}$

13. Show that the lines $\frac{x-5}{7} = \frac{y+2}{-5} = \frac{z}{1}$ and $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ are perpendicular to each other.

Ans. The equations of the given lines are $\frac{x-5}{7} = \frac{y+2}{-5} = \frac{z}{1}$ and $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$

The direction ratios of the given lines are 7, −5, 1 and 1, 2, 3 respectively.

Two lines with direction ratios, a_1, b_1, c_1 and a_2, b_2, c_2, are perpendicular to each other,

if $a_1a_2 + b_1 b_2 + c_1c_2 = 0$

$\therefore 7 \times 1 + (-5) \times 2 + 1 \times 3$

$= 7 - 10 + 3$

$= 0$

Therefore, the given lines are perpendicular to each other.

14. Find the shortest distance between the lines

$$\vec{r}=\left(\hat{i}+2\hat{j}+\hat{k}\right)+\lambda\left(\hat{i}-2\hat{j}+\hat{k}\right)\text{and}$$
$$\vec{r}=2\hat{i}-\hat{j}-\hat{k}+\mu\left(2\hat{i}+\hat{j}+2\hat{k}\right)$$

Ans. The equations of the given lines are

$$\vec{r}=\left(\hat{i}+2\hat{j}+\hat{k}\right)+\lambda\left(\hat{i}-2\hat{j}+\hat{k}\right)\text{and}$$
$$\vec{r}=2\hat{i}-\hat{j}-\hat{k}+\mu\left(2\hat{i}+\hat{j}+2\hat{k}\right)$$

It is known that the shortest distance between the lines, $\vec{r}=\vec{a}_1+\lambda\vec{b}_1\text{and}\vec{r}=\vec{a}_2+\mu\vec{b}_1$ is given by,

$$d=\left|\frac{\left(\vec{b}_1\times\vec{b}_2\right).\left(\vec{a}_2-\vec{a}_2\right)}{\left|\vec{b}_1\times\vec{b}_2\right|}\right|.........(1)$$

Comparing the given equations, we obtain

$$\vec{a}_1=\hat{i}+2\hat{j}+\hat{k}$$
$$\vec{b}_2=\hat{i}-\hat{j}+\hat{k}$$
$$\vec{a}_2=2\hat{i}-\hat{j}-\hat{k}$$
$$\vec{b}_2=2\hat{i}+\hat{j}+2\hat{k}$$
$$\vec{a}_2-\vec{a}_1=\left(2\hat{i}-\hat{j}-\hat{k}\right)-\left(\hat{i}+2\hat{j}+k\right)=\hat{i}-3\hat{j}-2\hat{k}$$

$$\vec{b}_1\times\vec{b}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & 1\\ 2 & 1 & 2\end{vmatrix}$$

$$\hat{b}_1\times\vec{b}_2=\left(-2-1\right)\hat{i}-\left(2-2\right)\hat{j}+\left(1+2\right)\hat{k}=-3\hat{i}+3\hat{k}$$

$$\Rightarrow\left|\vec{b}_1\times\vec{b}_2\right|=\sqrt{\left(-3\right)^2+\left(3\right)^2}=\sqrt{9\times9}=\sqrt{18}=3\sqrt{2}$$

Substituting all the values in equation (1), we obtain

$$d=\left|\frac{\left(-3\hat{i}+3\hat{k}\right).\left(\hat{i}-3\hat{j}-2\hat{k}\right)}{3\sqrt{2}}\right|$$

$$\Rightarrow d=\left|\frac{-3.1+3\left(-2\right)}{3\sqrt{2}}\right|$$

$$\Rightarrow d = \left|\frac{-9}{3\sqrt{2}}\right|$$

$$\Rightarrow d = \frac{3}{\sqrt{2}} = \frac{3\times\sqrt{2}}{\sqrt{2}\times\sqrt{2}} = \frac{3\sqrt{2}}{2}$$

Therefore, the shortest distance between the two lines is $\frac{3\sqrt{2}}{2}$ units.

15. Find the shortest distance between the lines $\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$ and $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$

Ans. The given lines are $\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$ and $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$

It is known that the shortest distance between the two lines,

$\frac{x+x_1}{a_1} = \frac{y+y_1}{b_1} = \frac{z+z_1}{c_1}$ and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$, is given by,

$$d = \frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{\left(b_1c_2-b_2c_1\right)^1 + \left(c_1a_2-a_2b_1\right)^1}}$$

Comparing the given equations, we obtain

$x_1 = -1, y_1 = -1, z_1 = -1$
$a_1 = 7, b_1 = -6, c_1 = 1$
$x_2 = 3, y_2 = 5, z_2 = 7$
$a_2 = 1, b_2 = -2, c_2 = 1$

then, $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} 4 & 6 & 8 \\ 7 & -6 & 1 \\ 1 & -2 & 1 \end{vmatrix}$

$=4(-6+2)-6(7-1)+8(-14+6)$
$=-6-36-64$
$=-16-36-64$
$=-116$

$$\Rightarrow\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}=\sqrt{(-6+2)^2+(1+7)^2+(-14+6)^2}$$

$=\sqrt{16+36+64}$
$=\sqrt{116}$
$=2\sqrt{29}$

Substituting all the values in equation (1), we obtain

$$d=\frac{-116}{2\sqrt{29}}=\frac{-58}{\sqrt{29}}=\frac{-2\times 29}{\sqrt{29}}=-2\sqrt{29}$$

Since distance is always non-negative, the distance between the given lines is $2\sqrt{29}$ units.

16. Find the shortest distance between the lines whose vector equations are

$$\vec{r}=(\hat{i}+2\hat{j}+3\hat{k})+\lambda(\hat{i}-3\hat{j}+2\hat{k})\text{and}\vec{r}=4\hat{i}+5\hat{j}+6\hat{k}+\mu(2\hat{i}+3\hat{j}+\hat{k})$$

Ans. The given lines are $\vec{r}=(\hat{i}+2\hat{j}+3\hat{k})+\lambda(\hat{i}-3\hat{j}+2\hat{k})\text{and}\vec{r}=4\hat{i}+5\hat{j}+6\hat{k}+\mu(2\hat{i}+3\hat{j}+\hat{k})$

It is known that the shortest distance between the lines, $\vec{r}=\vec{a}_1+\lambda\vec{b}_1\text{and}\vec{r}=\vec{a}_2+\mu\vec{b}_2$ is given by,

$$d=\left|\frac{(\vec{b}_1\times\vec{b}_2).(\vec{a}_2-\vec{a}_2)}{|\vec{b}_1\times\vec{b}_2|}\right|\text{.........}(1)$$

Comparing the given equations with $\vec{r}=\vec{a}_1+\lambda\vec{b}_1\text{and}\vec{r}=\vec{a}_2+\mu\vec{b}_2$, we obtain

$\vec{a}_1=\hat{i}+2\hat{j}+3\hat{k}$
$\vec{b}_2=\hat{i}-3\hat{j}+2\hat{k}$
$\vec{a}_2=4\hat{i}+5\hat{j}+6\hat{k}$
$\vec{b}_2=2\hat{i}+3\hat{j}+\hat{k}$
$\vec{a}_2-\vec{a}_1=(4\hat{i}+5\hat{j}+6\hat{k})-(\hat{i}+2\hat{j}+3\hat{k})=3\hat{i}+3\hat{j}+3\hat{k}$

$$\Rightarrow|\vec{b}_1\times\vec{b}_2|=\sqrt{(-9)^2+(3)^2+(9)^2}=\sqrt{81+9+81}=\sqrt{171}=3\sqrt{19}$$

$(\vec{b}_1\times\vec{b}_2).(\vec{a}_2-\vec{a}_1)=(-9\hat{i}+3\hat{j}+9\hat{k}).(3\hat{i}+3\hat{j}+3\hat{k})$
$=-9\times 3+3\times 3+9\times 3$
$=9$

Substituting all the values in equation (1), we obtain

$$d = \left|\frac{9}{3\sqrt{19}}\right| = \frac{3}{\sqrt{19}}$$

Therefore, the shortest distance between the two given lines is $\frac{3}{\sqrt{19}}$ units.

17. Find the shortest distance between the lines whose vector equations are

$$\vec{r} = (1-t)\hat{i} + (t-2)\hat{j} + (3-2t)\hat{k} \text{ and}$$
$$\vec{r} = (s+1)\hat{i} + (2s-1)\hat{j} + (2s+1)\hat{k}$$

Ans. The given lines are

$$\vec{r} = (1-t)\hat{i} + (t-2)\hat{j} + (3-2t)\hat{k}$$
$$\Rightarrow \vec{r} = (\hat{i} - 2\hat{j} + 3\hat{k}) + t(-\hat{i} + \hat{j} - 2\hat{k}) \ldots\ldots\ldots (1)$$
$$\vec{r} = (s+1)\hat{i} + (2s-1)\hat{j} - (2s+1)\hat{k}$$
$$\Rightarrow \vec{r} = (\hat{i} - \hat{j} + \hat{k}) + s(\hat{i} + 2\hat{j} - 2\hat{k}) \ldots\ldots\ldots (2)$$

It is known that the shortest distance between the lines $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1 \text{and} \vec{r} = \vec{a}_2 + \mu \vec{b}_2$ is given by,

$$d = \left|\frac{(\vec{b}_1 \times \vec{b}_2).(\vec{a}_2 - \vec{a}_2)}{|\vec{b}_1 \times \vec{b}_2|}\right|$$

For the given equations,

$$= \hat{i} - 2\hat{j} + 3\hat{k}$$
$$\vec{b}_1 = -\hat{i} - \hat{j} - 2\hat{k}$$
$$\vec{a}_2 = \hat{i} - \hat{j} - \hat{k}$$
$$\vec{a}_2 = \hat{i} + 2\hat{j} - 2\hat{k}$$
$$\vec{a}_2 - \vec{a}_1 = (\hat{i} - \hat{j} - \hat{k}) - (\hat{i} - 2\hat{j} + 3\hat{k}) = \hat{j} - 4\hat{k}$$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & -2 \\ 1 & 2 & -2 \end{vmatrix} = (-2+4)\hat{i} - (2+2)\hat{j} + (-2-1)\hat{k} = 2\hat{i} - 4\hat{j} - 3\hat{k}$$

$$\vec{a}_1 = \hat{i} - 2\hat{j} + 3\hat{k}$$
$$\vec{b}_1 = -\hat{i} + \hat{j} - 2\hat{k}$$
$$\vec{a}_2 = \hat{i} - \hat{j} - \hat{k}$$

$\vec{b}_2 = \hat{i} + 2\hat{j} - 2\hat{k}$

$\vec{a}_2 - \vec{a}_1 = (\hat{i} - \hat{j} - \hat{k}) - (\hat{i} - 2\hat{j} + 3\hat{k}) = \hat{j} - 4\hat{k}$

$\Rightarrow |\vec{b}_1 \times \vec{b}_2| = \sqrt{(2)^2 + (-4)^2 + (-3)^2} = \sqrt{4 + 16 + 9} = \sqrt{29}$

$\therefore (\vec{b}_1 \times \vec{b}_2).(\vec{a}_2 - \vec{a}_1) = (2\hat{i} - 4\hat{j} - 3\hat{k}).(\hat{j} - 4\hat{k}) = -4 + 12 = 8$

Substituting all the values in equation (3), we obtain

$$d = \left|\frac{8}{\sqrt{29}}\right| = \frac{8}{\sqrt{29}}$$

Therefore, the shortest distance between the lines is $\frac{8}{\sqrt{29}}$ units