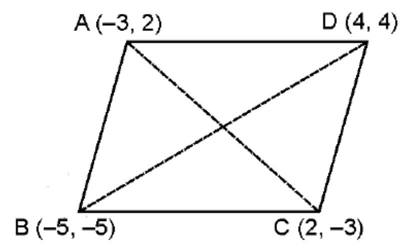
Class – 09	Topic – Distance and Section Formula

1. A $(-3, 2)$, B $(-5, -5)$, C $(2, -3)$ and D $(4, 4)$ are the four points in a plane. Show that ABCD is a rhombus but not a square.

Solution

The given points are A (–3, 2), B (–5, –5), C (2, –3) and D (4, 4).



$$\therefore \quad AB = \sqrt{[-5-(-3)]^2 + (-5-2)^2}$$
$$= \sqrt{(-5+3)^2 + (-7)^2}$$
$$= \sqrt{(-2)^2 + (-7)^2}$$
$$= \sqrt{4+49}$$
$$= \sqrt{53} \text{ units}$$

$$BC = \sqrt{[2-(-5)]^2 + [-3-(-5)]^2} = \sqrt{(2+5)^2 + (-3+5)^2}$$
$$= \sqrt{7^2 + 2^2} = \sqrt{49+4} = \sqrt{53} \text{ units}$$

$$CD = \sqrt{(4-2)^2 + [4-(-3]^2} = \sqrt{2^2 + (4+3)^2}$$
$$= \sqrt{2^2 + 7^2} = \sqrt{4+49} = \sqrt{53} \text{ units}$$

$$DA = \sqrt{(-3-4)^2 + (2-4)^2}$$
$$= \sqrt{(7)^2 + (-2)^2} = \sqrt{49+4} = \sqrt{53} \text{ units}$$

$\therefore$ AB = BC = CD = DA.

$\therefore$ ABCD is either a rhombus or a square.

$$\text{Diag. } AC = \sqrt{[2-(-3)]^2 + (-3-2)^2}$$
$$= \sqrt{(2+3)^2 + (-5)^2}$$
$$= \sqrt{5^2 + 5^2} = \sqrt{25+25} = \sqrt{50} = 5\sqrt{2} \text{ units.}$$

$$\text{Diag. } BD = \sqrt{(2+3)^2 + (-5)^2}$$

$$= \sqrt{[4-(-5)]^2 + [4-(-5)]^2}$$
$$= \sqrt{(4+5)^2 + (4+5)^2}$$
$$= \sqrt{9^2 + 9^2} = \sqrt{81+81}$$
$$= \sqrt{162} = 9\sqrt{2} \text{ units}$$

$\therefore$ Diag. AC $\neq$ Diag. BD

$\therefore$ ABCD is a rhombus but not a square. Proved

2. Find the co-ordinates of the circumcenter of ΔABC with vertices ar A (3, 0), B (-1, -6) and C(4, -1). Also, find its circum-radius.

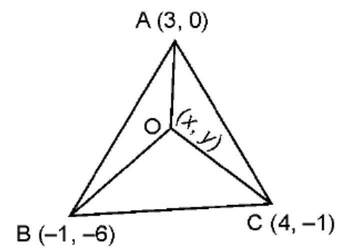
Solution The vertices of the given triangle are A (3, 0), B (–1, –6) and C (4, –1).

Let O (x, y) be the circumcentre of ΔABC.

Then,

$$OA = OB = OC$$

$$\Rightarrow OA^2 = OB^2 = OC^2$$



Now, $OA^2 = OB^2$

$$\Rightarrow (x-3)^2 + (y-0)^2 = [x-(-1)]^2 + [y-(-6)]^2$$

$$\Rightarrow (x-3)^2 + y^2 = (x+1)2 + (y+6)^2$$

$$\Rightarrow x^2 + y^2 - 6x + 9 = x^2 + y^2 + 2x + 12y + 37$$

$$\Rightarrow 8x + 12y + 28 = 0$$

$$\Rightarrow 2x + 3y = -7 \quad ...(i)$$

And, $OB^2 = OC^2$

$$\Rightarrow [x-(-1)]^2 + [y-(-6)]^2 = (x-4)^2 + [y-(-1)]^2$$

$$\Rightarrow (x+1)^2 + (y+6)^2 = (x-4)^2 + (y+1)^2$$

$$\Rightarrow x^2 + y^2 + 2x + 12y + 37 = x^2 + y^2 - 8x + 2y + 17$$

$$\Rightarrow 10x + 10y + 20 = 0$$

$$\Rightarrow x + y = -2 \quad ...(ii)$$

Solving (i) and (ii), we get $x = 1$ and $y = -3$.

∴ The circumcentre of ΔABC is O (1, –3).

$$\text{Circum-radius} = OA = \sqrt{(1-3)^2 + (-3-0)^2}$$

$$= \sqrt{(-2)^2 + (-3)^2}$$

$$= \sqrt{4+9}\ \sqrt{13}$$

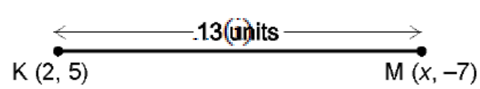
$$= \sqrt{13} \text{ units.}$$

3. KM is a straight line of 13 units. If K has the co-ordinates (2, 5) and M has the co-ordinates (x, -7), find the possible values of x.

Solution We have, $KM^2 = (x-2)^2 + (-7-5)^2$

$$\Rightarrow KM^2 = x^2 + 4 - 4x + 144$$

$$\Rightarrow KM^2 = x^2 - 4x + 148 \quad ...(i)$$



But KM = 13 (given)

$$\Rightarrow KM^2 = 169 \quad ...(ii)$$

∴ From (i) and (ii), we get

$$169 = x^2 - 4x + 148$$

$$\Rightarrow x^2 - 4x - 21 = 0$$

$$\Rightarrow x^2 - 7x + 3x - 21 = 0$$

$$\Rightarrow x(x-7) + 3(x-7) = 0$$

$$\Rightarrow (x-7)(x+3) = 0$$

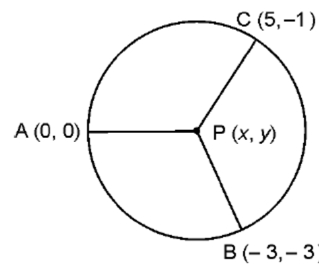
$$\Rightarrow x = 7 \text{ or } x = -3$$

Hence, the possible values of x are 7 and –3.

4. Find the co-ordinates of the centre of a circle which passes through the points A (0, 0), B (-3, 3) and C (5, -1). Also, find the radius of the circle.

Solution

Let P (x, y) be the centre of the circle passing through the points A (0, 0), B (–3, 3) and C (5, –1



Then, PA = PB = PC

$\Rightarrow$ $PA^2 = PB^2 = PC^2$

Now, $PA^2 = PB^2$

$\Rightarrow (x-0)^2 + (y-0)^2 = (x+3)^2 + (y-3)^2$

$\Rightarrow x^2 + y^2 = x^2 + 9 + 6x + y^2 + 9 - 6y$

$\Rightarrow 6x - 6y = -18$

$\Rightarrow x - y = -3$ ...(i)

And, $PB^2 = PC^2$

$\Rightarrow (x+3)^2 + (y-3)^2 = (i-5)^2 + (y+1)^2$

$\Rightarrow x^2 + 9 + 6x + y^2 + 9 - 6y = x^2 + 25 - 10x + y^2 + 1 + 2y$

$\Rightarrow 16x - 8y = 8$

$\Rightarrow 2x - y = 1$ ...(ii)

Subtracting (ii) from (i), we get

$-x = -4 \Rightarrow x = 4$

Substituting $x = 4$ in (i), we get

$4 - y = -3 \Rightarrow y = 7$

Hence, centre of the circle is P (4, 7).

Radius of the circle = PA

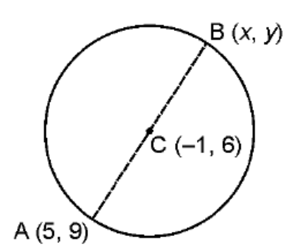
$$= \sqrt{(4-0)^2 + (7-0)^2}$$

$$= \sqrt{16+49} = \sqrt{65} \text{ units.}$$

5. The centre of a circle is C (-1, 6) and one end of a diameter is A (5, 9). Find the co-ordinates of the other end.

Solution

Let the other end of the diameter of the circle be B (x, y) whose one end is the point A (5, 9).



$\therefore$ The mid-point of AB is .

$$\left[\frac{5+x}{2}, \frac{9+y}{2}\right]$$

The centre of the circle is C (–1, 6).

Since the centre of the circle is the mid-point of AB,

$$\frac{5+x}{2} = -1 \quad \text{and} \quad \frac{9+y}{2} = 6$$

$\Rightarrow 5 + x = -2$ and $9 + y = 12$

$\Rightarrow x = -7$ and $y = 3$

6. Find the distance between the following pairs of points.
   (i) $A(-2,5)$ and $B(3,-7)$
   (ii) $A(4,5)$ and $B(-3,2)$
   (iii) $A(-4,-4)$ and $B(3,5)$

Solution

(i) Using the distance formula $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ we get

$AB = \sqrt{\left(3 - (-2)\right)^2 + (-7 - 5)^2}$

$= \sqrt{5^2 + 12^2} = \sqrt{25 + 44} = \sqrt{169} = 13$ units

(ii) Using the distance formula $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ we get

$AB = \sqrt{(-3 - 4)^2 + (2 - 5)^2} = \sqrt{(-7)^2 + (-3)^2} = \sqrt{49 + 9} = \sqrt{58}$ units

(iii) We have $AB = \sqrt{\left(3 - (-4)\right)^2 + \left(5 - (-4)\right)^2} = \sqrt{7^2 + 9^2} = \sqrt{49 + 81} =$

$\sqrt{130}$ units

7. By using the distance formula, prove that each of the following sets of points are vertices of a right triangle:
   (i) $(4,4), (3,5), (-1,-1)$
   (ii) $(12,8), (-2,6), (6,0)$

Solution

(i) Let the points $(4, 4), (3, 5), (-1, -1)$ represent the points A, B and C, respectively. Then, $AB^2 = (3 - 4)^2 + (5 - 4)^2 = 1^2 + 1^2 = 2$

$BC^2 = (-1 - 3)^2 + (-1 - 5)^2 = (-4)^2 + (-6)^2 = 52$

and $CA^2 = \left(4 - (-1)\right)^2 + \left(4 - (-1)\right)^2 = 5^2 + 5^2 = 50$

Since $BC^2 = AB^2 + CA^2$, it follows from the converse of the Pythagorean Theorem that the triangle ABC is a right triangle, with right angle at A.

(ii) Let the points $(12, 8), (-2$, and $6), (6, 0)$ represent the points A, B and C, respectively. Then $AB^2 = (-2 - 12)^2 + (6 - 8)^2 = (-14)^2 + (-2)^2 = 200$

$BC^2 = \left(6 - (-2)\right)^2 + (0 - 6)^2 = 8^2 + (-6)^2 = 100$

and $CA^2 = (12 - 6)^2 + (8 - 0)^2 = 6^2 + (8)^2 = 100$

Since $AB^2 = BC^2 + CA^2$, it follows the converse of the Pythagorean Theorem in which the triangle ABC is a right triangle with right angle at C.

8. Show that the triangle with vertices $(4, 3), (7, -1)$ and $(9, 3)$ is an isosceles triangle.

Solution

Let the points $(4, 3), (7, -1), (9, 3)$ represent the points A, B and C, respectively.

Then, $AB^2 = (7 - 4)^2 + (-1 - 3)^2 = 3^2 + (-4)^2 = 25,$

$BC^2 = (9 - 7)^2 + \left(3 - (-1)\right)^2 = 2^2 + 4^2 = 20,$

and $CA^2 = (9 - 4)^2 + (3 - 3)^2 = 5^2 + 0^2 = 25$

Since $AB^2 = CA^2$, we get $AB = CA$.

Therefore, the triangle ABC is an isosceles triangle.

9. Show that the triangle with vertices $(a, a), (-a, -a)$ and $\left(-a\sqrt{3}, a\sqrt{3}\right)$ is an equilateral triangle.

Solution

Let the points $(a, a), (-a, -a)$ and $\left(-a\sqrt{3}, a\sqrt{3}\right)$ represent the points A, B and C, respectively.

Then $AB^2 = (-a - a)^2 + (-a - a)^2$

$= (-2a)^2 + (-2a)^2 = 4a^2 + 4a^2 = 8a^2,$

$BC^2 = \left(-a\sqrt{3} - (-a)\right)^2 + \left(a\sqrt{3} - (-a)\right)^2$

$= \left(-a\sqrt{3} + a\right)^2 + \left(a\sqrt{3} + a\right)^2$

$= 2\left\{\left(-a\sqrt{3}\right)^2 + a^2\right\}$ $\quad [\therefore (a - b)^2 + (a + b)^2 = 2(a^2 + b^2)]$

$= 2(3a^2 + a^2) = 8a^2$

and $CA^2 = \left(a - \left(-a\sqrt{3}\right)\right)^2 + \left(a - a\sqrt{3}\right)^2 = \left(a + a\sqrt{3}\right)^2 + \left(a - a\sqrt{3}\right)^2$

$= 2\left\{a^2 + \left(a\sqrt{3}\right)^2\right\}$ $\quad [\therefore (a + b)^2 + (a - b)^2 = 2(a^2 + b^2)]$

$= 2(a^2 + 3a^2) = 8a^2.$

Since, $AB^2 = BC^2 = CA^2$, we get $AB = BC = CA$.

Thus, the triangle ABC is an equilateral triangle.

10. Find the value of x such that AB = BC, where A, B and C are (6, −1), (1, 3) and (x, 8), respectively.

Solution

A (6, −1), B (1, 3) and C (x, 8) such that AB = BC.

We have $AB^2 = (1-6)^2 + (3-(-1))^2 = (-5)^2 + 4^2 = 41$.

and $BC^2 = (x-1)^2 + (8-3)^2 = (x-1)^2 + 25$

Since $AB = BC$, we have $AB^2 = BC^2$

$\Rightarrow (x-1)^2 + 25 = 41$

$\Rightarrow (x-1)^2 = 16$

$\Rightarrow x - 1 = \pm 4$

$\Rightarrow x = 1 \pm 4$

Therefore, $x = 5, -3$.

11. Which point on the x-axis is equidistant from (7, 6) and (−3, 4).

Solution

Let us take (7, 6) and (−3, 4) to be points A and B respectively. Let C $(x, 0)$ on x-axis be the point which is equidistant from A and B, that is AC = BC. This implies

$AC^2 = BC^2$

$\Rightarrow (x-7)^2 + (0-6)^2 = (x-(-3))^2 + (0-4)^2$

$\Rightarrow x^2 - 14x + 49 + 36 = x^2 + 6x + 9 + 16$

$\Rightarrow -14x + 85 = 6x + 25$

$\Rightarrow 60 = 20x$

$\Rightarrow x = 3$.

Thus, the required point is (3,0).

12. An equilateral triangle has one vertex at the point (3, 4) and another at (−2, 3). Find the coordinates of the third vertex.

Solution

Let us take the two given vertices to be A (3, 4) and B (−2, 3). Let the third vertex of the equilateral triangle be C(x, y). Then AB = BC = CA. This implies$AB^2 = BC^2\ CA^2$.

Since $BC^2 = CA^2$, we get

$(x + 2)^2 + (y - 3)^2 = (x - 3)^2 + (y - 4)^2$

$x^2 + 4x + 4 + y^2 - 6y + 9 = x^2 - 6x + 9 + y^2 - 8y + 16$

$10x + 2y - 12 = 0$

$5x + y - 6 = 0 \qquad \ldots(1)$

Next, since $AB^2 = BC^2$, we get

$(-2 - 3)^2 + (3 - 4)^2 = (x + 2)^2 + (y - 3)^2$

$\Rightarrow 25 + (-1)^2 = (x + 2)^2 + (y - 3)^2$

$\Rightarrow 26 = (x + 2)^2 + (y - 3)^2 \qquad \ldots(2)$

From (1) we have $y = 6 - 5x$. Putting this in (2), we get

$(x + 2)^2 + (6 - 5x - 3)^2 = 26$

$(x + 2)^2 + (3 - 5x)^2 = 26$

$x^2 + 4x + 4 + 9 - 30x + 25x^2 = 26$

$\Rightarrow 26x^2 - 26x - 13 = 0 \Rightarrow 2x^2 - 2x - 1 = 0$

$$\Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{4 - 4(2)(-1)}}{2(2)}$$

$$= \frac{2 \pm \sqrt{12}}{4} = \frac{2 \pm 2\sqrt{3}}{4} = \frac{1 \pm \sqrt{3}}{2}$$

When $x = \frac{1 + \sqrt{3}}{2}, y = 6 - 5\left(\frac{1 + \sqrt{3}}{2}\right) = \frac{12 - 5 - 5\sqrt{3}}{2} = \frac{1}{2}(7 - 5\sqrt{3})$

When $x = \frac{1 - \sqrt{3}}{2}, y = 6 - 5\left(\frac{1 - \sqrt{3}}{2}\right) = \frac{12 - 5 + 5\sqrt{3}}{2} = \frac{1}{2}(7 + 5\sqrt{3})$

Therefore, the coordinates of C are

$$\left(\frac{1}{2}(1 + \sqrt{3}), \frac{1}{2}(7 - 5\sqrt{3})\right) \text{ or } \left(\frac{1}{2}(1 - \sqrt{3}), \frac{1}{2}(7 + 5\sqrt{3})\right)$$

13. Find the abscissa of points whose ordinate is 4 and which are at a distance of 5 from (5, 0).

Solution

Let the abscissa of the required point be x. Then A (x, 4) is at a distance of 5 from B(5, 0). That is AB =

$\Rightarrow AB^2 = 5^2$

$\Rightarrow (x - 5)^2 + (4 - 0)^2 = 25$

$\Rightarrow (x - 5)^2 = 25 - 16 = 9$

$\Rightarrow x - 5 = \pm 3$

$\Rightarrow x = 5 \pm 3$

$\Rightarrow x = 2$ or 8.

14. Find the relation that must exist between x and y, so that C (x, y) is equidistant from A (6, −1) and B (2, 3).

Solution

Since $CA = CB$, we have $CA^2 = CB^2$

$\Rightarrow (x - 6)^2 + (y + 1)^2 = (x - 2)^2 + (y - 3)^2$

$\Rightarrow (x - 6)^2 - (x - 2)^2 = (y - 3)^2 - (y + 1)^2$

$\Rightarrow (x - 6 + x - 2)(x - 6 - x + 2) = (y - 3 + y + 1)(y - 3 - y - 1)$

$\Rightarrow (2x - 8)(-4) = (2y - 2)(-4)$

$\Rightarrow 2x - 8 = 2y - 2$

$x - 4 = y - 1$ or $x - y = 3$ is the relation between x and y.

15. Find the circumcenter of the triangle whose angular points are A (4, 3), B (−2, 3) and C (6, −1).

Solution

A (4, 3), B (−2, 3) and C (6, −1) are the vertices of the triangle ABC.

Let D(x, y) be the circumcenter of the triangle ABC. By definition

$DA = DB = DC \Rightarrow DA^2 = DB^2 = DC^2$

$\Rightarrow (x - 4)^2 + (y - 3)^2 = (x + 2)^2 + (y - 3)^2$

$= (x - 6)^2 + (y + 1)^2 \quad \ldots (1)$

The first two relations give us

$(x - 4)^2 = (x + 2)^2 \Rightarrow x - 4 = \pm(x + 2)$

But $x - 4 = x + 2$ leads to the absurd relation $-4 = 2$, and $x - 4 = -(x + 2)$ gives us $2x = 2$ or $x = 1$.

From the last two relations in (1), we get

$(x + 2)^2 + (y - 3)^2 = (x - 6)^2 + (y + 1)^2$

$\Rightarrow (1 + 2)^2 - (1 - 6)^2 = (y + 1)^2 - (y - 3)^2$ [$\because x = 1$]

$\Rightarrow 9 - 25 = (y + 1 + y - 3)(y + 1 - y + 3)$

$\Rightarrow -16 = (2y - 2)(4) = 8y - 8$

$-8 = 8y$ or $y = -1$.

Thus, the required circumcenter is (1, −1)