Board –CBSE | Class –10th | Topic – Real Number

**(1) Euclid's Division Lemma:**

Theorem: Given positive integers a and $b$, there exist unique integers $q$ and $r$ satisfying $a = bq + r, 0 \leq r < b$

**(2) Euclid's division algorithm:**

To obtain the HCF of two positive integers, say c and d, with c $> d$, follow the steps below:

Step 1: Apply Euclid's division lemma, to c and d. So, we find whole numbers, q, and $r$ such that $c = dq + r, 0 \leq r < d$

Step 2: If $r = 0$, $d$ is the HCF of $c$ and $d$. If $r \neq 0$, apply the division lemma to $d$ and $r$.

Step 3: Continue the process till the remainder is zero. The divisor at this stage will be the required HCF.

Example: Use Euclid's division algorithm to find the HCF of 135 and 225 .

Step 1: Here 225 $>$ 135, on applying the division lemma to 225 and 135 ,

we get $225 = 135\times1 + 90$

Step 2: Since, remainder $\neq 0$, we again apply division lemma to 135 and 90 ,

we get $135 = 90\times1 + 45$

Step 3: Again, applying division lemma to 90 and 45, we get $90 = 45 \times 2 + 0$

The remainder has become zero. And since the divisor at this step is 45 , the HCF of 135 and 225 is 45.

**(3) The Fundamental Theorem of Arithmetic**

Theorem; Every composite number can be expressed (factorized) as a product of primes, and this factorization is unique, apart from the order in which the prime factors occur. In general, given a composite number $x$, we factorise it as $x = p_1 p_2 \ldots p_n$, where $p_1, p_2, \ldots, p_n$ are primes and written in ascending order, i.e., $p_1 \leq p_2 \leq \ldots \leq p_n$. If we combine the same primes, we will get powers of primes.

For Example: The prime factors of 32760 $= 2\times2\times2\times3\times3\times5\times7\times13 = 2^3 \times 3^2\times5\times7\times13$

For Example Find the HCF and LCM of 96 and 404 by the prime factorization method.

The prime factorization of 96 is $2^5\times3$. And that of 404 is $2^2\times101$.

Hence, HCF of 96 and 404 will be $2^2 = 4$.

Now, LCM $(96, 404) = (96\times404)/(HCF(96, 404)) = (96\times404)/4 = 9696$

**(4) Revisiting Irrational Numbers:**

Irrational Number: Irrational numbers are the numbers that cannot be written in $p/q$ form, where $p$ and $q$ are integers and $q\neq0$.

**Theorem 1:** Let $p$ be a prime number. If p divides $a^2$, then $p$ divides a, where a is a positive integer.

Proof: Suppose the prime factorization of a is as follows:

(i) $a = p_1p_{2...} ... p_n$, where $p_1, p_{2,...}p_n$ are primes.

(ii) On squaring both the sides, we get,

(iii) $a^2 = \left(p_1p_2 ... p_n\right)\left(p_1p_2 ... p_n\right) = p_1^2.... p_n^2$

(iv) It is given that p divides $a^2$. Hence, we can say that $p$ is one of the prime factors of $a^2$ as per the Fundamental Theorem of Arithmetic.

(v) However, as per the uniqueness part of the Fundamental Theorem of Arithmetic, we can deduce that the only prime factors of $a^2$ are $p_1p_{2....}p_{n-}$ Thus, $p$ is one of $p_1p_2 ... p_m$.

Since, $a = p_1, p_2 ... p_n, p$ divides a.

**Theorem 2:** $\sqrt{2}$ is irrational.

Proof: We shall start by assuming $\sqrt{2}$ as rational. In other words, we need to find integers $x$ and y such that $\sqrt{2} = x/y$

(i) Let $x$ and y have a common factor other than 1, and so we can divide by that common factor and assume that $x$ and $y$ are co-prime. So, $y\sqrt{2} = x$

(ii) Squaring both sides, we get, $2y^2 = x^2$.

(iii) Thus, 2 divides $x^2$ and by theorem, we can say that 2 divides $x$.

(iv) Hence, $x = 2z$ for some integer $z$.

(v) Substituting $x$, we get, $2x^2 = 4z^2 e{\cdot}y^2 = 4z^2$; which means $y^2$ is divisible by 2, and so $y$ will also be divisible by 2.

(vi) Now, from theorem, $x$ and $y$ will have 2 as a common factor. But, it is opposite to the fact

that $x$ and $y$ are co-prime.

(vii) Hence, we can conclude $\sqrt{2}$ is irrational.

Let us assume $6 + \sqrt{2}$ to be rational.

Therefore, we must find two integers $a,\ b\ (b \neq 0)$ such that $6 + \sqrt{2} = \frac{a}{b}$ i. e. $\sqrt{2} = \frac{a}{b} - 6$

Since a and b are integers, $a/b - 6$ is also rational and hence $\sqrt{2}$ must be rational.

Now, this contradicts the fact that $\sqrt{2}$ is irrational.

Hence, $6 + \sqrt{2}$ is irrational.

**(5) Revisiting Rational Numbers and Their Decimal Expansions:**

Theorem 1: Let $x$ be a rational number whose decimal expansion terminates. Then $x$ can be expressed in the form, p/q where p and q are co-prime, and the prime factorization of $q$ is of the form $2^n5^m$, where $n, m$ are non-negative integers.

Example: $13/125 = 13/5^3 = \left(13{\times}2^3\right)/\left(2^3 \times 5^3\right) = 104/10^3 = 0.104$

Theorem 2: Let $x = p/q$ be a rational number, such that the prime factorization of $q$ is of the form $2^n5^m$, where n and m are non-negative integers. Then $x$ has a decimal expansion that terminates.

Theorem 3; Let $x = p/q$ be a rational number, such that the prime factorization of $q$ is not of the form $2^n 5^m$, where n and m are non-negative integers. Then, $x$ has a decimal expansion which is non-terminating repeating (recurring).

Example: Without actually performing the long division, state whether 6/15 will have a terminating decimal expansion or a non-terminating repeating decimal expansion.

The prime factorization of 6/15 can be written as

$6/15 = (2\times3)/(3\times5) = 2/5$

Here, the denominator is of the form $5^n$.

Hence, the decimal expansion of 6/15 is terminating.

Example 1: Show that any positive odd integer is of the form $4q + 1$ or $4q + 3$, where $q$ is some integer.

Let a be any positive odd integer. And we apply the division algorithm with a and $b = 4$. As $0 \leq r < 4$, the possible remainders could be 0, 1, 2 and 3.

So, a can be $4q$, or $4q + 1$, or $4q + 2$, or $4q + 3$, where $q$ is the quotient.

Now, since a is odd, so a cannot be $4q$ or $4q + 2$ (as both are divisible by 2 ).

Hence, any odd integer is of the form $4q + 1$ or $4q + 3$.

Example 2: Check whether $6^n$ can end with the digit 0 for any natural number $n$.

If a number ends with digit 0, then, it must be divisible by 10, or in other words, it will be divisible by 2 and 5 as $10 = 2 \times 5$

Now, prime factorization of $6^n = (2 \times 3)^n$

Here, 5 is not in the prime factorization of $6^n$. Hence, for any value of $n$, $6^n$ will not be divisible by 5. Thus, $6^n$ cannot end with the digit 0 for any natural number $n$.